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# Principles of Automatic Control
## -Chap3 Dynamic Response(1)


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# Chap3 Dynamic Response

**3.1 Transfer Function**

**3.2 Block Diagram**

**3.3 Signal-Flow Graph Models and Mason’s rule**

**3.4 Control System Stability Analysis**

**3.5 Control System Steady-state characteristics**

**3.6 Control System Dynamic characteristics**

**(Three performance indexes for Automatic Control System)**

# 3.1 Transfer Function

**Transfer function: the basic and the most important concepts for classical automatic control .**

**The transfer function of a linear, stationary system is defined as the radio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero.**

**If Input--r(t)，output--c(t). Transfer function was define as：**

$$H(s)=\frac{L[c(t)]}{L[r(t)]}=\frac{C(s)}{R(s)}$$

**where, C(s)=L[c(t)]——Laplace transform for output**

**R(s)=L[r(t)]——Laplace transform for input**

**So, we can get:**

$$C(s)=R(s)H(s)$$

**The time response of control system c(t) equals the inverse Laplace transform of C(s) ：**

$$c(t)=[C(s)]=L^{-1}[R(s)H(s)]$$

# 3.1 Transfer Function

**In general, the time domain mathematical model of the system-Differential Equation is:**

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots\cdots + a_1 \frac{dc(t)}{dt} + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots\cdots + b_1 \frac{dr(t)}{dt} + b_0 r(t)$$

**where,** $a_i, b_j$ **(i =0,1,2,…….n; j =0,1,2…….m) are real number, which is determined by the system structure parameters. The Laplace transformation are used to both sides :**

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots\cdots a_1 sC(s) + a_0 sC(s)$$

$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots\cdots b_1 sR(s) + b_0 R(s)$$

**So, the control system transfer function of general expression:**

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots\cdots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots\cdots a_1 s + a_0}$$

$$G(s) = \frac{C(s)}{R(s)} = K_r \frac{(s+z_1)(s+z_2)\cdots\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots\cdots(s+p_n)}$$

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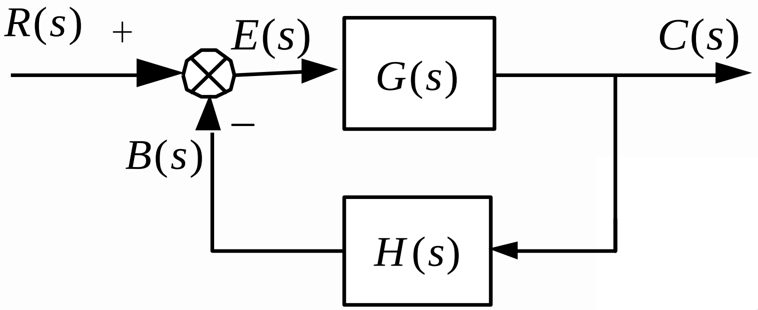
# 3.1 Transfer Function

## Transfer Functions of Systems

### 1. Open loop transfer function




**Definition：the ratio of feedback signal to error signal**

$$\frac{B(s)}{E(s)} = G(s)H(s)$$

**Conclusion：** Open loop transfer function is equal the transfer function G(s) in forward path time to transfer function H(s) in feedback path.

# 3.1 Transfer Function

## The general situation：

$$G(s)H(s)=\frac{b_m s^m + b_{m-1}s^{m-1} + \cdots\cdots b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots\cdots a_1 s + a_0}$$

$$=\frac{K\prod_{i=1}^{u}(\tau_i s+1)\prod_{i=1}^{\eta}(\tau_{di}^2 + 2\zeta_{di}\tau_d s+1)}{s^{v}\prod_{i=1}^{\rho}(T_i s+1)\prod_{i=1}^{\sigma}(T_{ni}^{i} s^2 + 2\zeta_{ni}T_{ni}s+1)}$$

**Where, K- the open loop amplification coefficient for closed-loop system (also called the open loop magnification or open-loop gain), is the important parameter to influence the system performance.**

**when the feedback transfer function H（s）=1, the open loop transfer function is equal to forward transfer function, that is G( s ).**

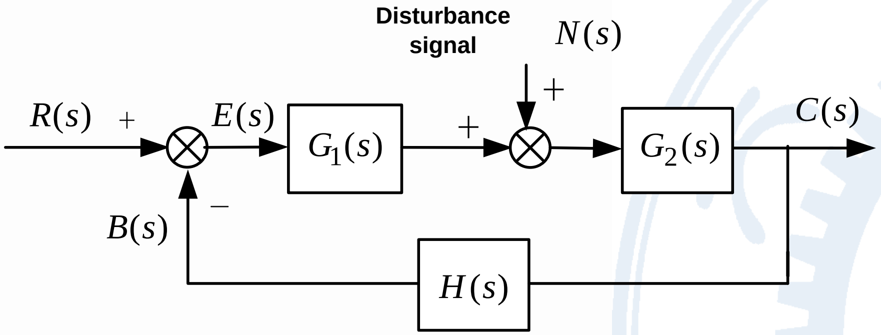
# 3.1 Transfer Function

## 2. Closed-loop Transfer Function

**Definition: the transfer function of the output and input when main feedback loop is connected, usually represented by Φ(s).**

$$\Phi(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

## 3. Disturbance Transfer Function




**Disturbance signal : outside of the system input function.**

# 3.1 Transfer Function

**When input R（s）=0**

$$\Phi_N(s)=\frac{C_N(s)}{N(s)}=\frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}$$

**we hypothesis :**

$$\left|G_1(s)G_2(s)H(s)\right|>>1 \qquad \left|G_1(s)H(s)\right|>>1 \quad \Rightarrow \quad \frac{C_N(s)}{N(s)}\to 0$$

**The disturbances signals can be restrained.**

**If disturbance signal N（s）=0**

$$\Phi_R(s)=\frac{C_R(s)}{R(s)}=\frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)}$$

**we hypothesis :** $\left|G_1(s)G_2(s)H(s)\right|>>1 \quad \Rightarrow \quad \frac{C_R(s)}{R(s)}\approx\frac{1}{H(s)}$

**It shows that the transfer function of closed-loop system only has relationship with H (S) on, is not depend on** $G_1(s), G_2(s)$

# 3.1 Transfer Function

**When R（s）、 N（s）are not equal to zero, the output C(s) is defined as:**

$$
\begin{aligned}
C(s) &= C_R(s) + C_N(s) \\
&= \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)}R(s) + \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}N(s) \\
&= \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}[G_1(s)R(s) + N(s)]
\end{aligned}
$$

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# 3.1 Transfer function

## 5. Error Transfer function

### a) Error transfer function with reference signal

**If N(s)＝0, then**

$$\frac{E(s)}{R(s)}=\frac{R(s)-C(s)H(s)}{R(s)}=1-\frac{C(s)H(s)}{R(s)}$$

$$=1-\frac{G_1(s)G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}=\frac{1}{1+G_1(s)G_2(s)H(s)}$$

**that is so-called Error Transfer function.**

# 3.1 Transfer function

## b) Error transfer function with disturbance signal：

$$\frac{E(s)}{N(s)}=\frac{-G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}$$

## c) Total Error with R(s) and N(s)

$$E(s)=\frac{1}{1+G_1(s)G_2(s)H(s)}R(s)-\frac{G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}N(s)$$

# 3.1 Transfer Function

## Summary for Transfer Function：

1) the transfer function is the system mathematical model (or link) in the complex number field, is the system characteristics description, reflect the linear time-invariant input of the system and output.

2) the transfer function depends only on the structural parameters of the system itself, and have no relationship with system outside input .

3) the transfer function is the complex variable S of the rational real fraction function, that is m≤n ( m, n are the highest order times of molecular and denominator)

4) If the input for the unit pulse function， that is r(t)=δ(t), R(s)=L[r(t)]=1, so

$$c(t) = L^{-1}[R(s)G(s)] = L^{-1}[G(s)]$$

This shows that at this time of the system C(t) and transfer function G (s) have single value corresponding relation, they can be used to characterize dynamic behavior of the system.

5) closed loop system transfer function G (s) and make the denominator for 0, that is the characteristics of the system equation.

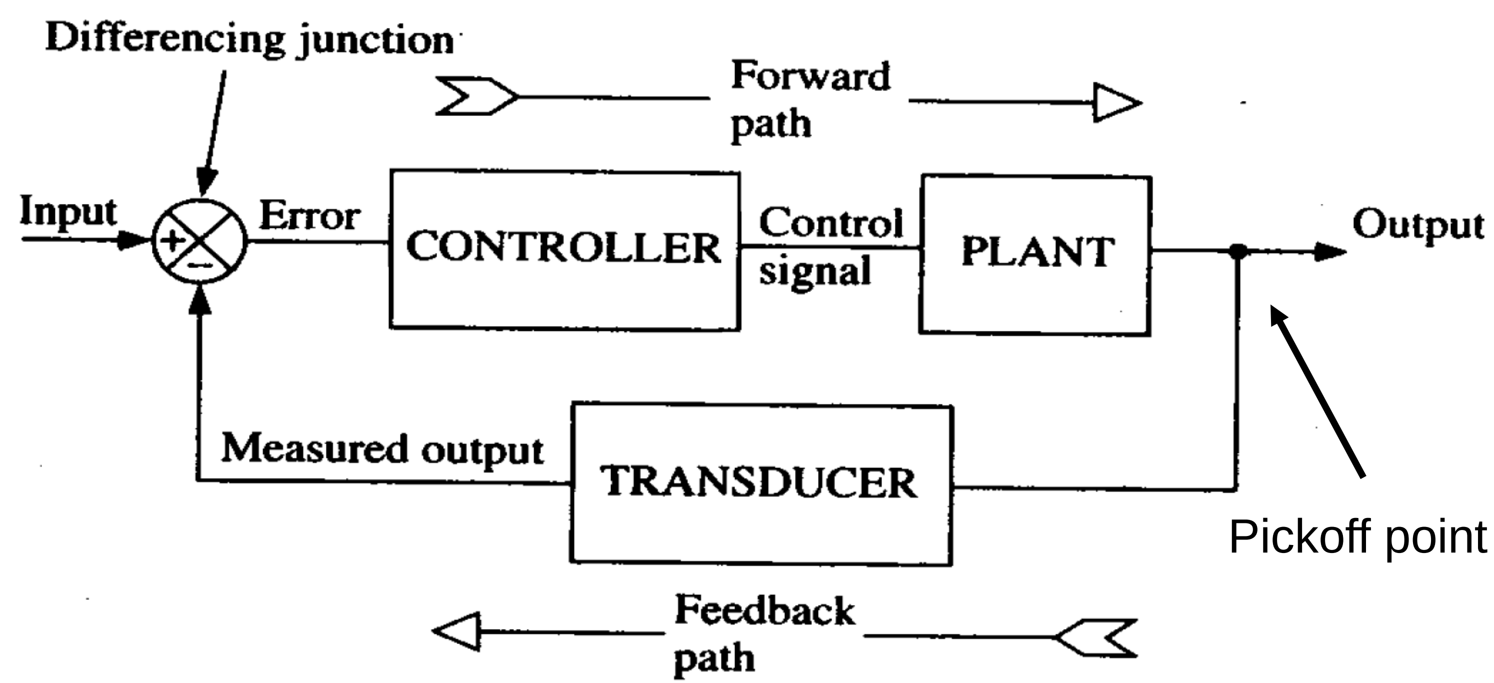
# 3.2 Block Diagram

- Graph model- Block Diagram
- Graph model have prominent advantage
  - Intuitive
  - Vivid
- Graph model is the most important means in automatic control (AC)
- Graph model  is always used to analyze complex system in engineering.

# 3.2 Block Diagram

## The Block Diagram Model

**– which consists of block, arrow, Differencing junction and pickoff point.**



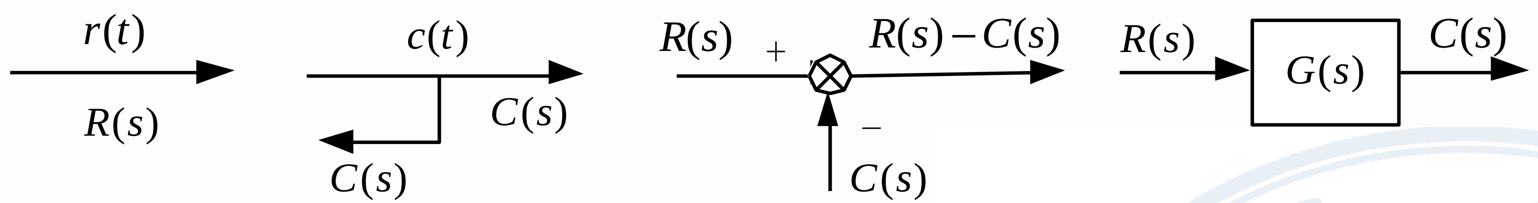

Generalized feedback control system

# 3.2 Block Diagram Model

- **A block diagram represents the flow of information and the function performed by each component in the system.**
- **Arrows are used to show the direction of the flow of information.**
- **The block represents the function or dynamic characteristics of the component and is represented by a transfer function.**
- **The complete block diagram shows how the functional components are connected and the mathematic equations that determine the response of each component.**

# 3.2 Block Diagram

## 1. Basic Element of Block Diagram



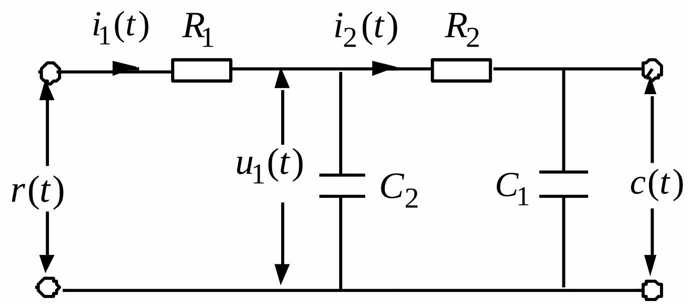

(a) (b) (c) (d)

**(a) Signal lines； (b) Branch point (also called pickoff point)；**
**(c) comparison points (also called summing point)； (d) block；**

**The system block diagram combine diagram and mathematical equations together, it describe the comprehensive characteristics for a system.**

# 3.2 Block Diagram

EX:

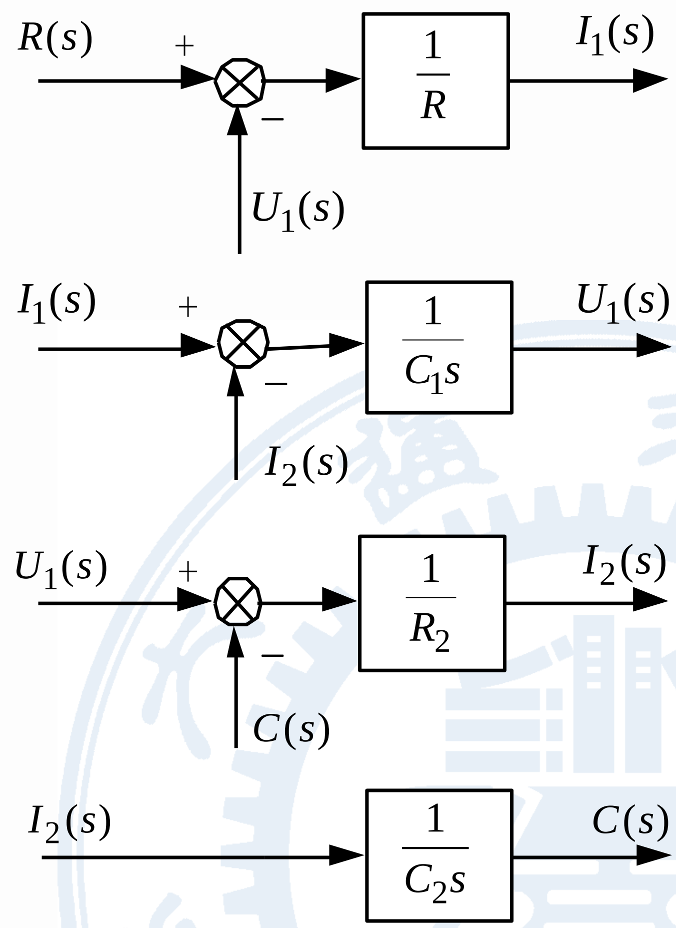



$$\frac{r(t)-u_1(t)}{R_1}=i_1(t)$$

$$u_1(t)=\frac{1}{C_1}\int[i_1(t)-i_2(t)]dt$$

$$\frac{u_1(t)-c(t)}{R_2}=i_2(t)$$

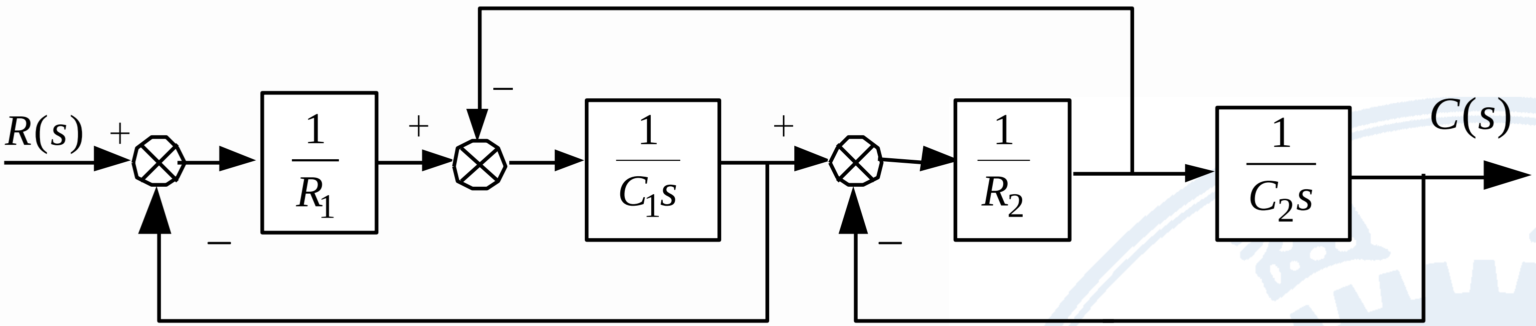
$$c(t)=\frac{1}{C_2}\int i_2(t)dt$$




(b)

# 3.2 Block Diagram

Assemble all block diagrams above：

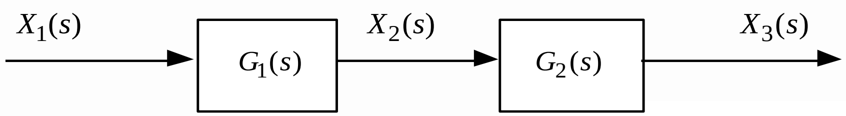



How can simplify this diagrams?

# 3.2 Block Diagram

## 2. Algorithm of Block Diagram：

### (1). series principle




## Proof:

$$G_1(s) = \frac{X_2(s)}{X_1(s)} \qquad G_2(s) = \frac{X_3(s)}{X_2(s)}$$

$$G(s) = \frac{X_3(s)}{X_1(s)} = \frac{X_2(s)}{X_1(s)} \bullet \frac{X_3(s)}{X_2(s)} = G_1(s)G_2(s)$$

**conclusion：** **G(s) is equal to the total series every link of the transfer function of the product** $G(s) = G_1(s)\, G_2(s) \ldots\ldots G_n(s)$

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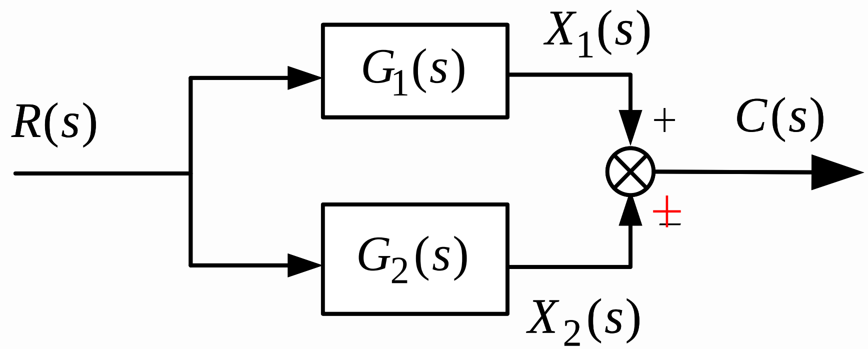
# 3.2Block Diagram

## (2). Parallel principle




**Proof:**

$$G_1(s)=\frac{X_1(s)}{R(s)} \qquad G_2(s)=\frac{X_2(s)}{R(s)} \qquad X_1(s)+X_2(s)=C(s)$$

**Is any questions or errors?**

$$G(s)=\frac{C(s)}{R(s)}=\frac{X_1(s)+X_2(s)}{R(s)}=\frac{X_1(s)}{R(s)}+\frac{X_2(s)}{R(s)}$$

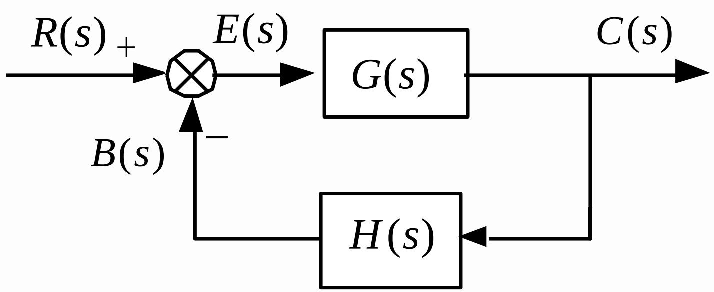
$$=G_1(s)+G_2(s)$$

**conclusion**： the transfer function of parallel link is equal to the sum of all the transfer function .

$$G(s) = G_1(s) + G_2(s) + \ldots\ldots + G_n(s)$$

# 3.2 Block Diagram

## 3、Feedback principle




**Here the two blocks are connected in a feedback arrangement so that each feeds into each other. When the feedback B(s) is subtracted, we call it Negative feedback.**

**Note: negative feedback is usually required for system stability.**

**Proof:**

$$C(s) = G(s)E(s) = G(s)[R(s) - B(s)]$$
$$= G(s)[R(s) - H(s)C(s)]$$

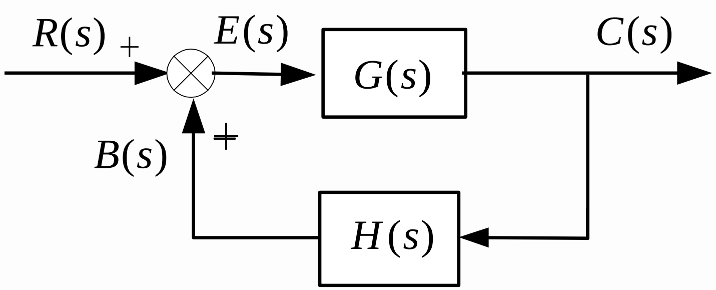
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The transfer function for Negative feedback

**Conclusion：The gain of a single-loop negative feedback system is given by the forward gain divided by the sum of 1 plus the loop gain.**

# 3.2 Block Diagram

When the feedback is added instead of subtracted, we call it **Positive feedback**. In this case, the gain is given by the forward gain divided by the sum of 1 minus the loop gain.




$$C(s) = G(s)E(s) = G(s)[R(s) + B(s)]$$
$$= G(s)[R(s) + H(s)C(s)]$$

$$C(s) = G(s)[R(s) + H(s)C(s)]$$

$$C(s) - G(s)H(s)C(s) = G(s)R(s)$$

$$C(s)[1 - G(s)H(s)] = G(s)R(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

The transfer function for Positive feedback

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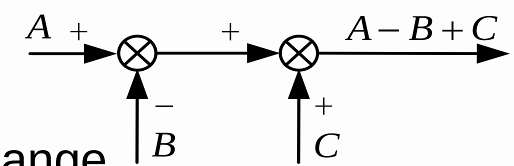
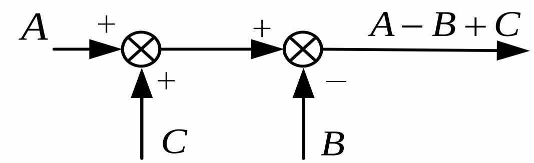
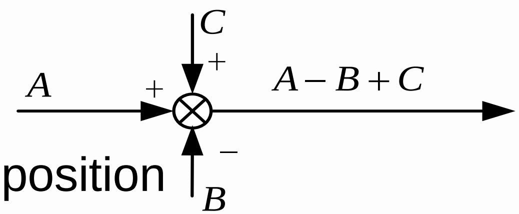
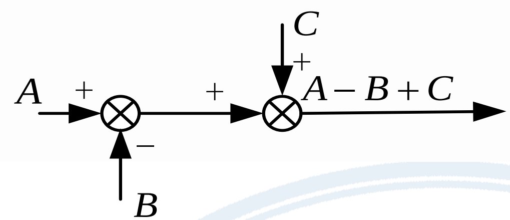
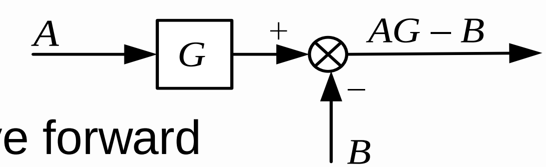
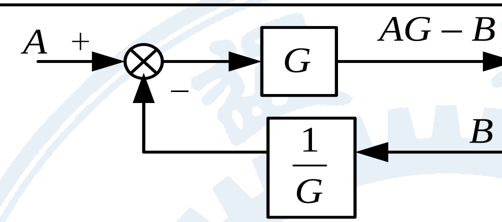
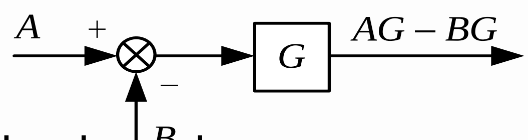
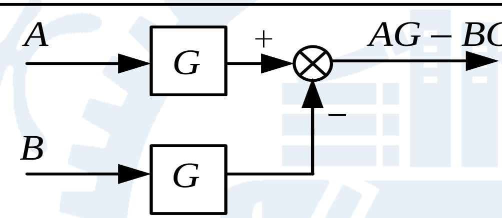
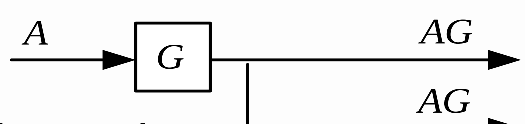
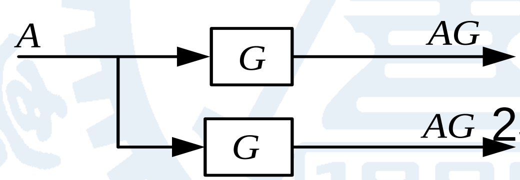
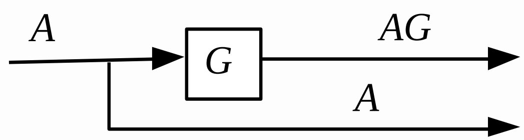
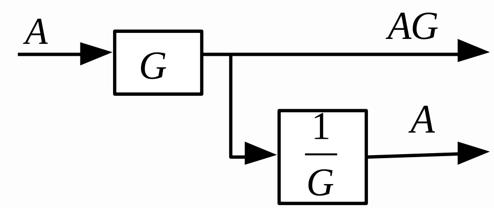
| 序号 | 变换方式 | 原方块图 | 等效方块图 |
|---|---|---|---|
| 1 | 比较点交换 comparison point exchange |   |   |
| 2 | 比较点分解 comparison point decomposition |   |   |
| 3 | 比较点前移 comparison point move forward |   |   |
| 4 | 比较点后移 comparison point move backward |   |   |
| 5 | 分支点前移 Pickoff point moved forward |   |   |

6 分支点后移

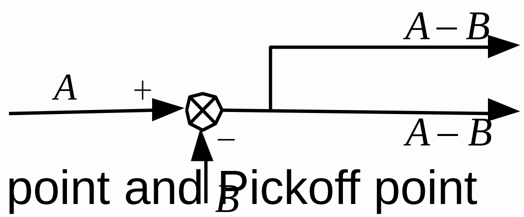
Pickoff point moved backward

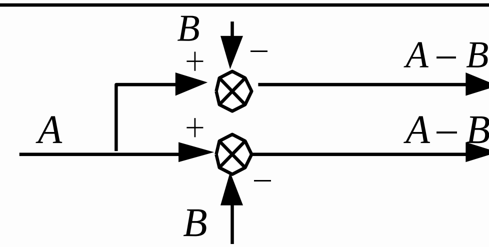



7 比较点与分支点交换

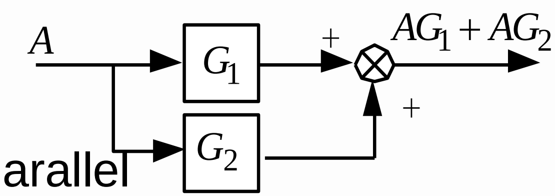
Exchange comparison point and Pickoff point

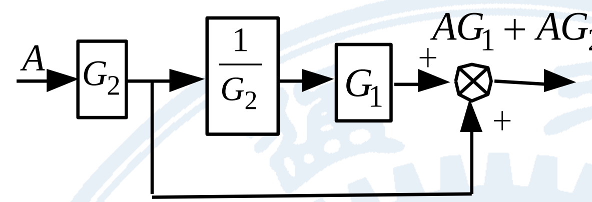



8 化成单位并联

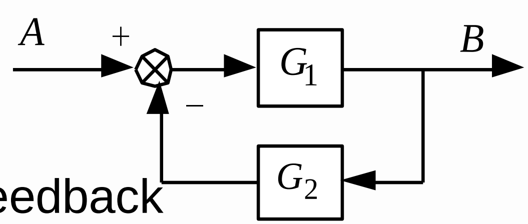
Transform into units Parallel

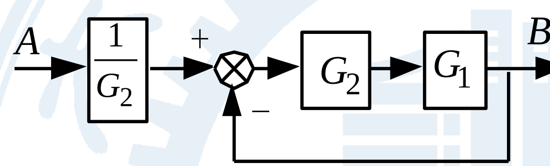



9 化成单位反馈

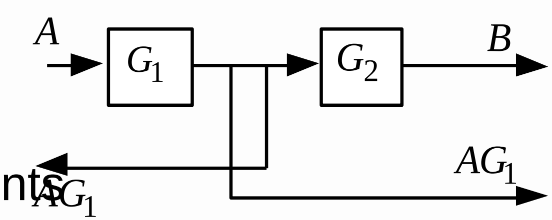
Transform into units feedback

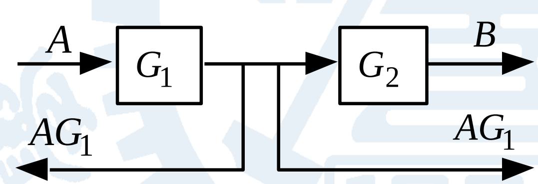



10 分支点交换

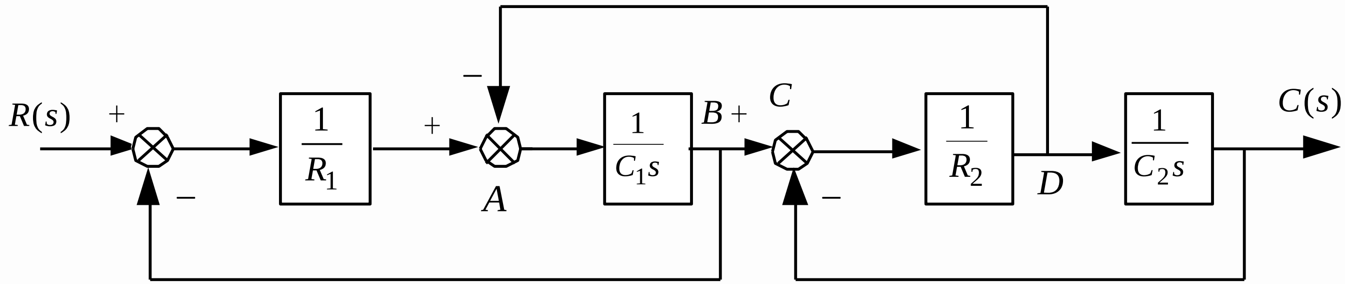
Exchange Pickoff points



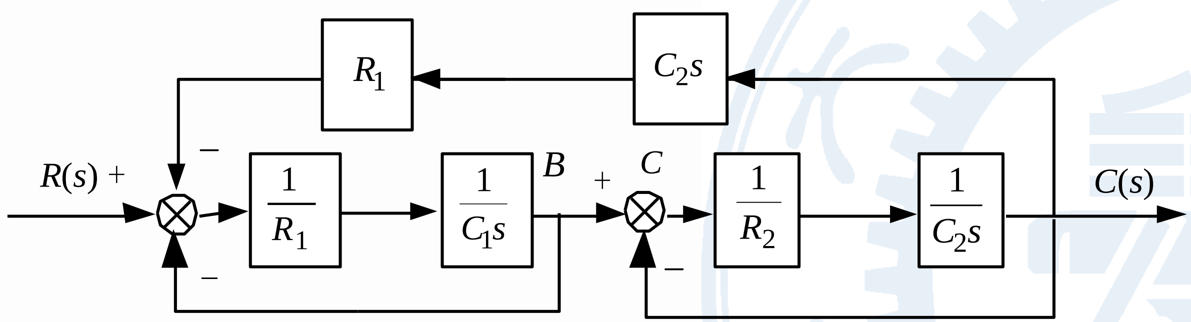
# 3.2 Block Diagram

**EX**




## Solution：Employ Block Diagram Algorithms

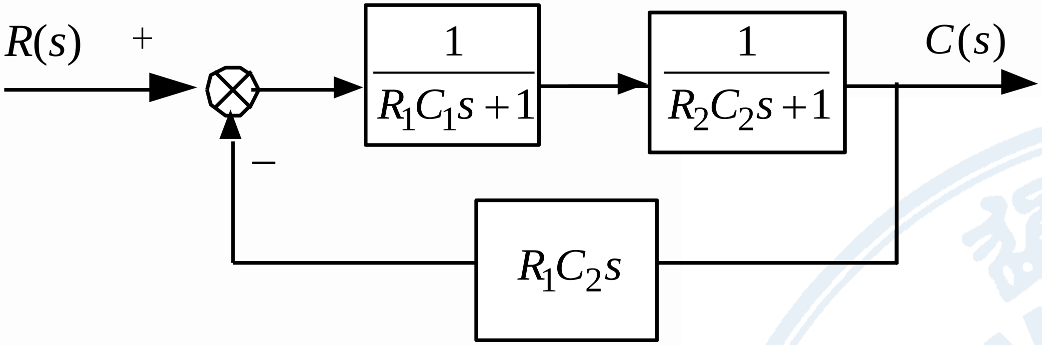
**(a) move comparison point A forwards, pickoff point D afterwards**




(*a*)

# 3.2 Block Diagram

## (b) Eliminate local feedback loop

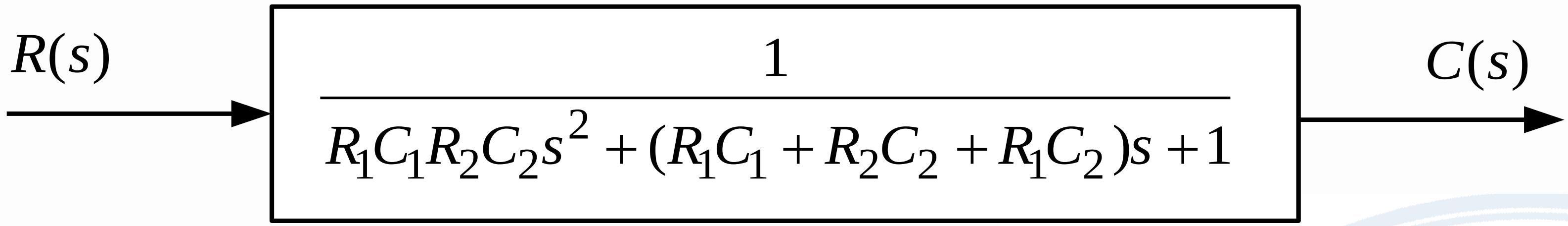



(b)

**Note: There are three local feedback loops in this examples**

# 3.2 Block Diagram

## （C） Eliminate main feedback loop and get the result




Conclusion: The simplified method for block diagram is not unique, we should make full use of all kinds of transformation skills, and choose the simple path, in order to achieve the simplest purposes.

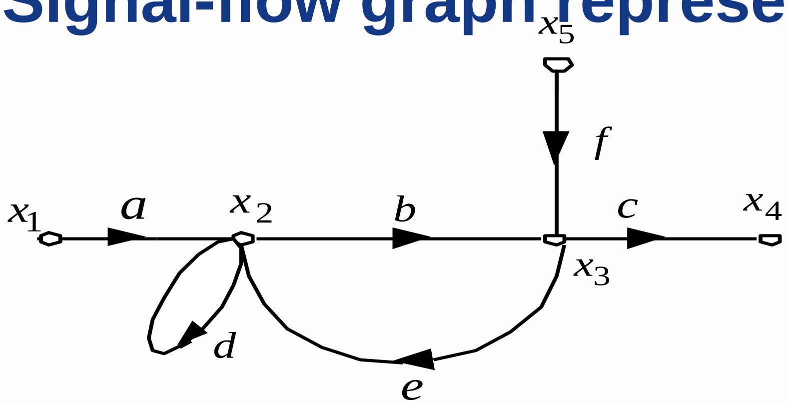
# 3.3 Signal-Flow Graph Models

**Signal-flow graph is a graphical representations of a set of linear algebraic equations.**

**Consider the following set of algebraic equations：**

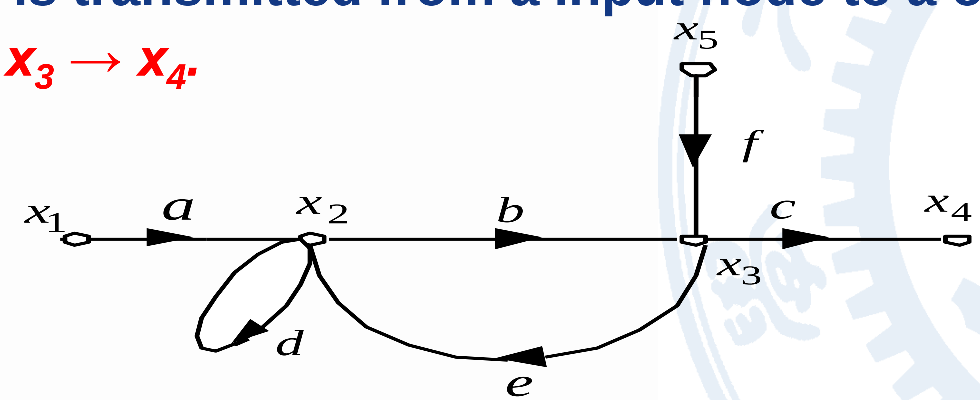
$$
\begin{cases}
x_1 = x_1 \\
x_2 = ax_1 + dx_2 + ex_3 \\
x_3 = bx_2 + fx_5 \\
x_4 = cx_3 \\
x_5 = x_5
\end{cases}
$$

**Signal-flow graph representation ：**

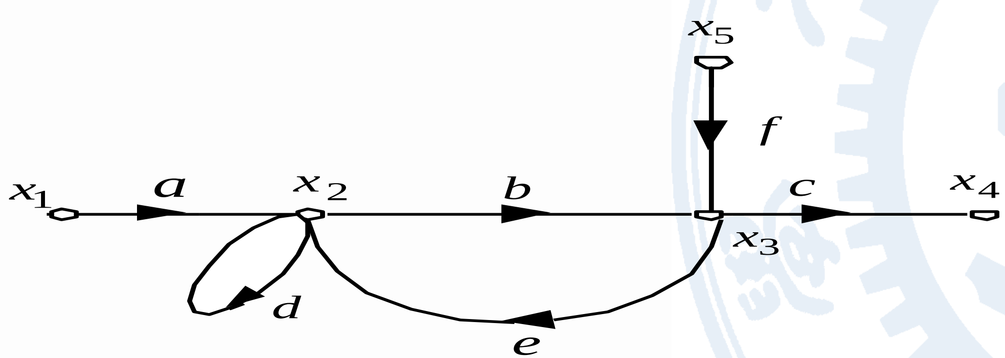


# 1. Definitions

- **Input node (or source node)【输入节点或源节点】:nodes have output branchs only,such as $x_1$, $x_5$.**
- **Output node (or sink node)【输出节点或阱节点】: nodes have input branches only, such as $x_4$.**
- **Mixed Node【混合节点】: nodes have both output and input branchs, another branch of the node type, such as $x_2$ , $x_3$.**
- **Transmission【传输】: the gain between two nodes. For example: the gain between $x_1 \rightarrow x_2$ is a, then the transmission is a.**
- **Forward path【前向通路】: the path that pass each node only once, when a signal is transmitted from a input node to a output node. Such as: $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4$.**



- **Overall Gain of forward path【前向通路总增益】：the gain product of each branch on forward path, Example: overall gain of $x_1 \to x_2 \to x_3 \to x_4$ is $abc$.**
- **Loop【回路】： a closed path that originates and terminates on the same nodes, and no node is met twice along the path.**
- **Loop Gain【回路增益】: the gain product of each branch of the loop. There are two loop in the graph, one is $x_2 \to x_3 \to x_2$ and the loop gain is $be$, the other is $x_2 \to x_2$, also known as self-loop, whose gain is $d$.**
- **Nontouching Loops【不接触回路】: loops that have no common node with each other. There is no such loops in the following graph.**

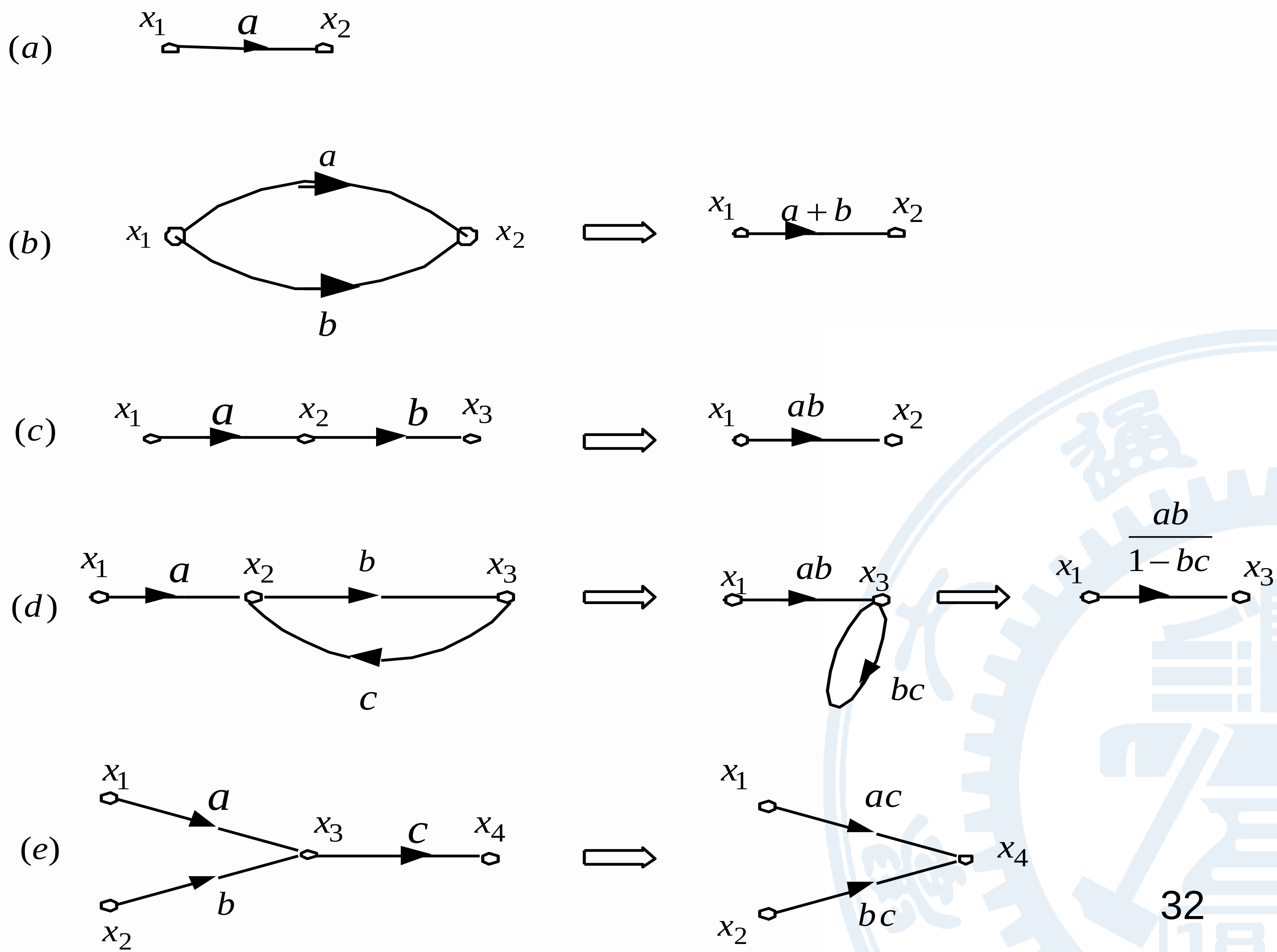


## 2. Properties and Algorithms of Signal-flow Graphs

### Properties:

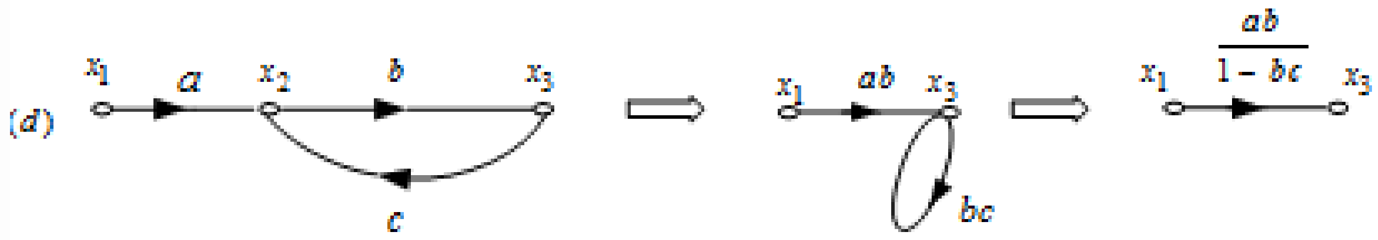
- 1. **Each node** represents a variable, and transmit the accumulation of all input signals to each output branch.
- 2. **A branch represents** the functional relationship between one signal and another. The direction of the arrow on the branch represents the flow direction of the signal.
- 3. **Mixed nodes** will become output nodes by increasing a branch with the gain of 1, and two ends of this branch represent the same variables.

# Algorithms for Signal-Flow Graph



# 3.3 Signal-Flow Graph Models

**Example: Make a simple derivation of the signal-flow in graph (d):**

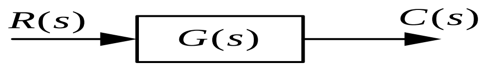
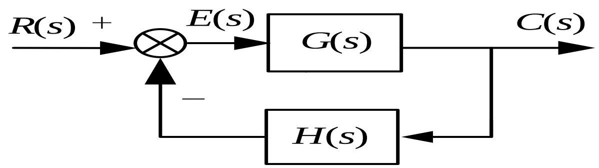
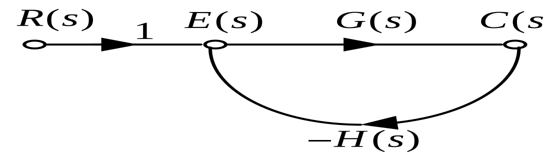
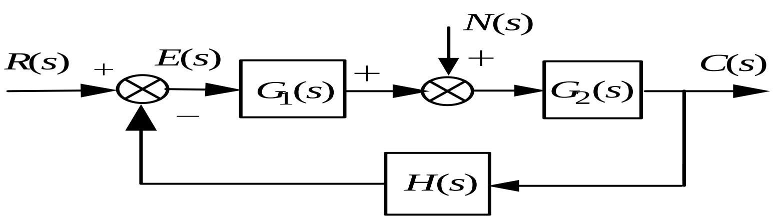
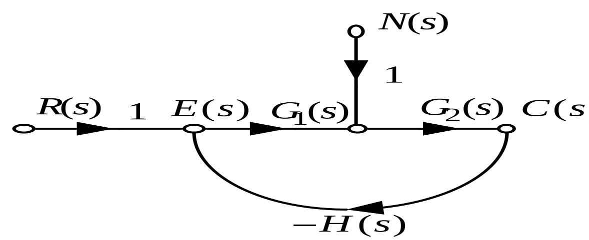
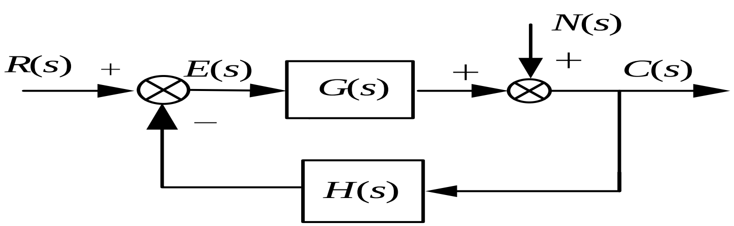
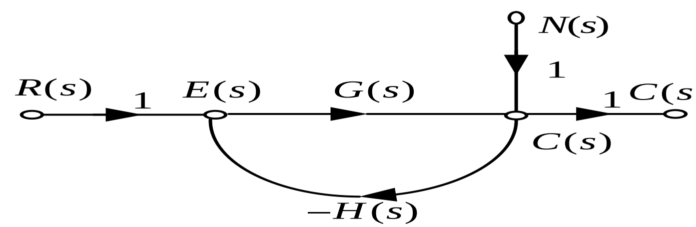
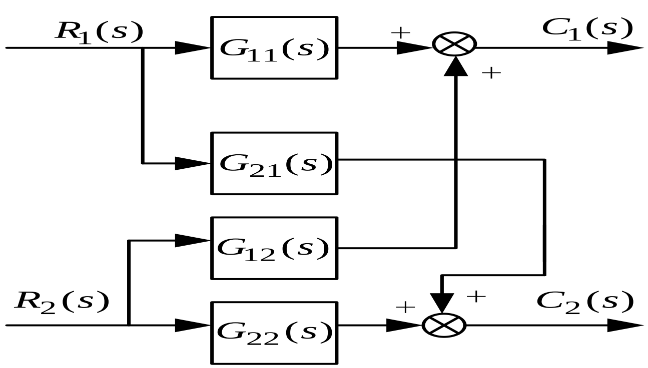
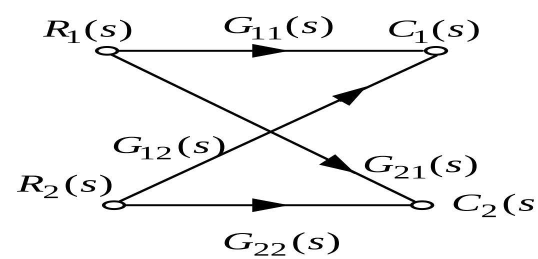



**since:** $x_2 = ax_1 + cx_3$

$x_3 = bx_2$

**eliminate intermediate variable x2, we have:**

$$x_3 = \frac{ab}{1 - bc} x_1$$

| 序号 | 方块图<br>Block Diagram | 信号流程图<br>Signal-Flow Graph |
|---|---|---|
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |
| 5 |  |  |

# 3.3 Signal-Flow Graph Models

## 3. Mason's rule (Mason Formula)

**The overall transmission (or overall gain) between input node and output node could be determined by Mason formula:**

$$G = \frac{1}{\Delta}\sum_{k=1}^{N} p_k \Delta_k$$

**Where:**

**Δ = the system determinant.**

**Δ = 1 - (sum of all individual loop gains)**

**+ (sum of the gain products of all combinations of two nontouching loops)**

**- (sum of the gain products of all combinations of three nontouching loops)**

**+ ……** $=1-\sum_m L_{m1} + \sum_m L_{m2} - \sum_m L_{m3} + \cdots\cdots$

$P_k$ **= gain of the *kth* forward path ;**

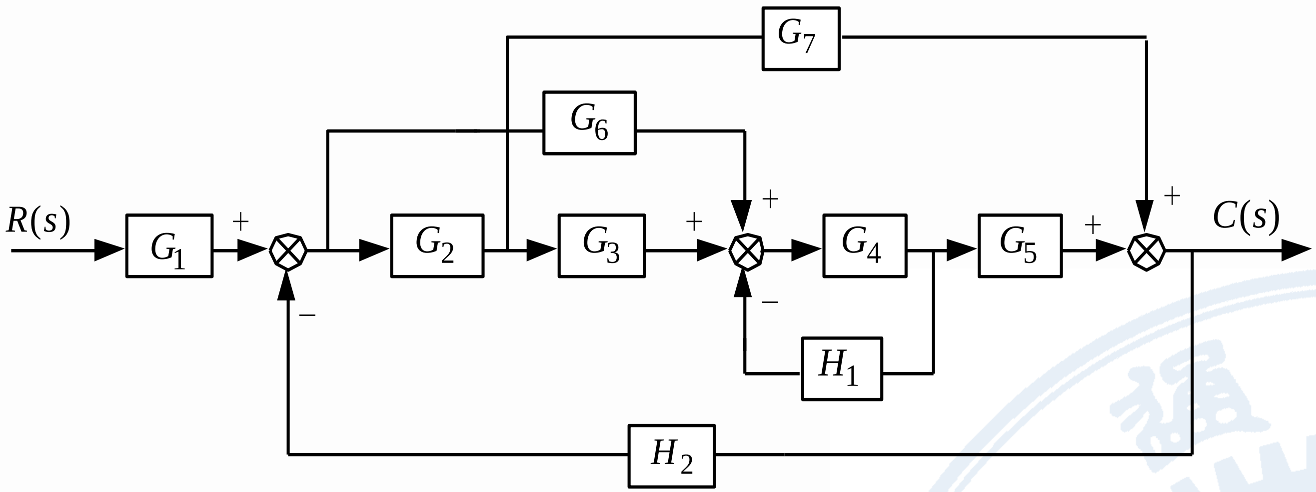
$L_{mr}$ **= gain product of *m* kinds of combinations in *r* nontouching loops**

**N = the total number of forward path;**

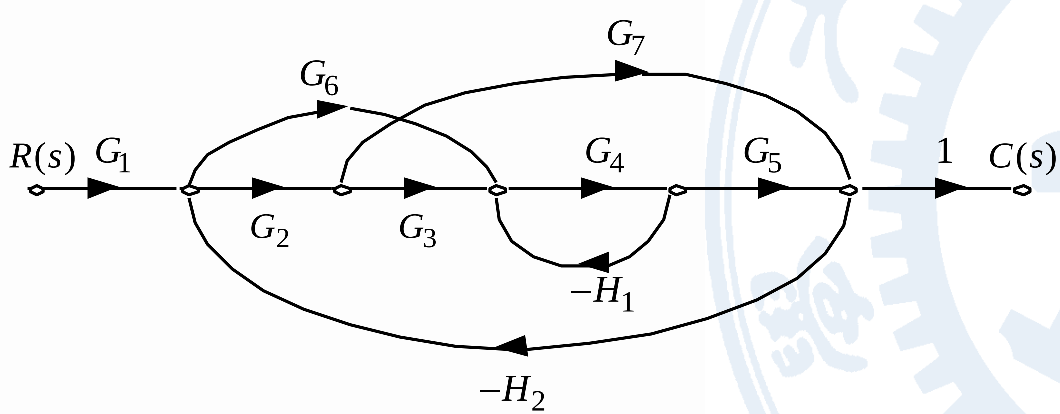
**$\Delta_k$ = cofactor of the *kth* path;**

# 3.3 Signal-Flow Graph Models

## EX1 Determine C(s) / R(s) by using Mason formula.




## Solution：Plot the signal-flow graph of the system



# 3.3 Signal-Flow Graph Models

**There are 4 independent loops in this graph：**

$L_1 = -G_4H_1$ $\qquad$ $L_2 = -G_2G_7H_2$

$L_3 = -G_6G_4G_5H_2$ $\qquad$ $L_4 = -G_2G_3G_4G_5H_2$

**The only nontouching loops are $L_1\ L_2$**

**hence, the determinat is**

$$\Delta=1-(L_1 + L_2 + L_3 + L_4) + L_1 L_2$$

**The 3 forward paths are：**

$P_1= G_1G_2G_3G_4G_5$ $\qquad$ $\Delta_1=1$

$P_2= G_1L_6G_4G_5$ $\qquad$ $\Delta_2=1$

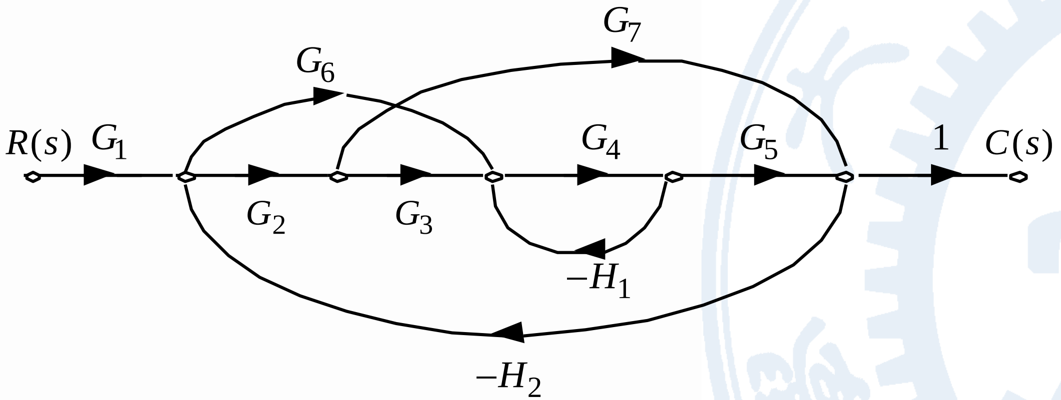
$P_3= G_1G_2G_7$ $\qquad$ $\Delta_3=1-L_1$

# 3.3 Signal-Flow Graph Models

**hence，the close-loop transfer function C(s) / R(s) is**

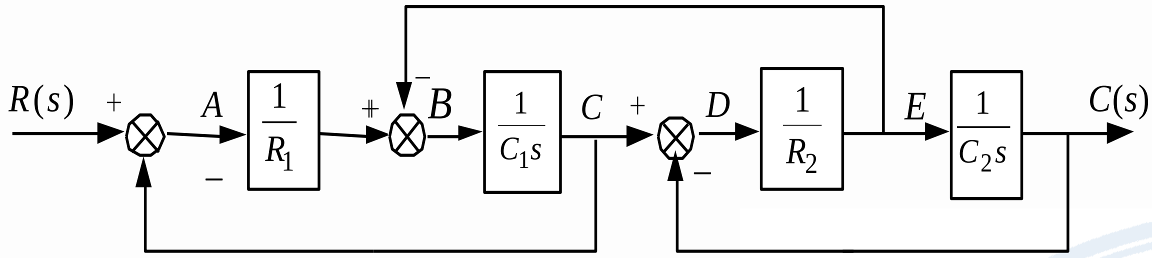
$$\frac{C(s)}{R(s)} = G = \frac{1}{\Delta}(p_1\Delta_1 + p_2\Delta_2 + p_3\Delta_3)$$

$$= \frac{G_1G_2G_3G_4G_5 + G_1G_6G_4G_3 + G_1G_2G_7(1+G_4H_1)}{1+G_4H_1 + G_2G_7H_2 + G_6G_4G_5H_2 + G_2G_3G_4G_5H_2 + G_4H_1G_2G_7H_2}$$

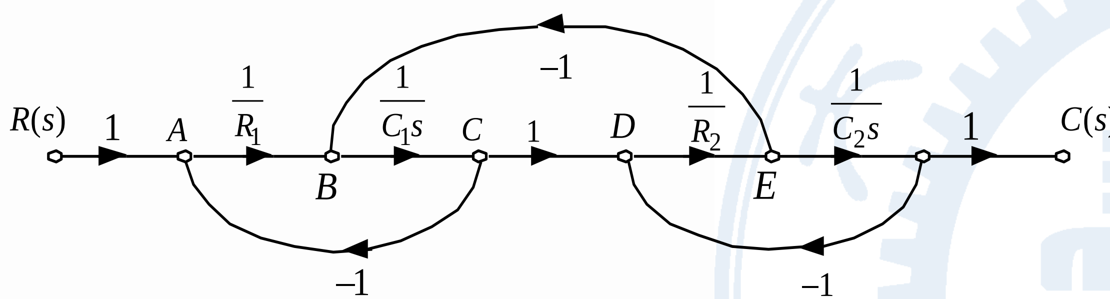


# 3.3 Signal-Flow Graph Models

## EX2： Determine C(s) / R(s) by using Mason formula.




- **Solution：Plot the signal-flow graph of the system**



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# 3.3 Signal-Flow Graph Models

**NOTES：node C is in front of comparison node D, in order to obtain output signal of node C, we need a branch with gain 1 to separate signals of C and D.**

**There are 3 independent loops $L_1$,$L_2$ and $L_3$. Nontouching loops are $L_1L_2$ :**

$$L_1=\frac{-1}{R_1C_1s}\quad L_2=\frac{-1}{R_2C_2s}\quad L_3=\frac{-1}{R_2C_1s}\quad L_1L_2=\frac{1}{R_1C_1sR_2C_2s}$$

$$\Delta=1-(L_1+L_2+L_3)+L_1L_2$$

$$=1+\frac{1}{R_1C_1s}+\frac{1}{R_2C_2s}+\frac{1}{R_2C_1s}+\frac{1}{R_1C_1R_2C_2s}$$

**There is only one forward path** $P_1=\frac{-1}{R_1R_2C_1C_2s^2}$ $\Delta_1=1$

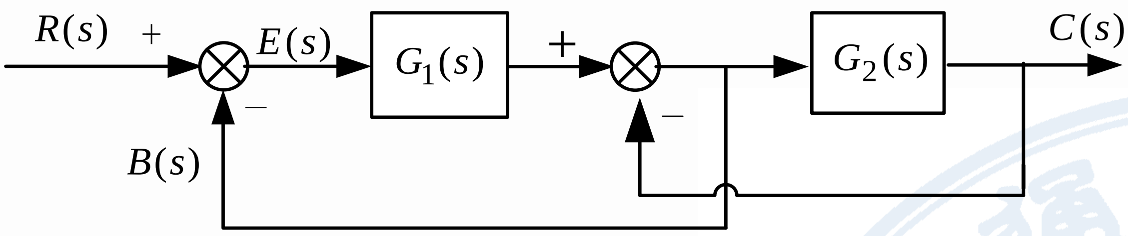
**hence,**

$$\frac{C(s)}{R(s)}=G=\frac{P_1\Delta_1}{\Delta}=\frac{1}{R_1R_2C_1C_2s^2+R_1C_1s+R_1C_2s+1}$$

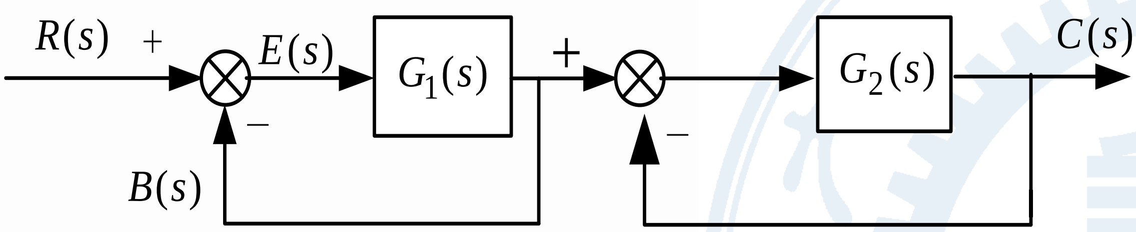
# 3.3 Signal-Flow Graph Models

**Please write the C(s) / R(s) by using Mason formula in classroom**

**EX1:**




**EX2:**



# Chap.3 Dynamic Response

- Review  for this chapter with textbook in Chap3
- NOTE:

This chapter is important for this automatic  control principles

Homework is useful for understand these content.

# Chap.3 Dynamic Response

## Homework(3)

- **P117,118:**
  - Ex.3.19 3.20
- **P125,126**
  - Ex.3.46 3.47

- **Deadline: Sep.29,2012**

**NOTE: Sep.26.2012, Exercise class(Q/A) , Mr. Xu and Mr.Cai will attendance this class.**

# Chapter 3 Dynamic Response

## -3.4 Control System Stability Analysis

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**Tel:021-34206192**
**Mobile:13918459696**

# Chapter 3 Dynamic Response

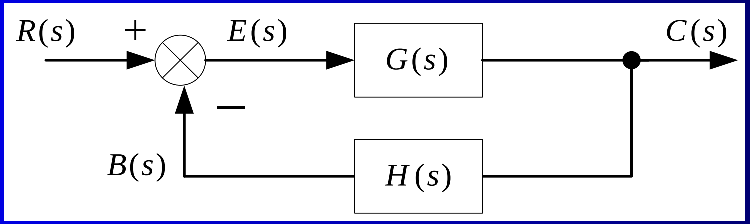
**Three performance indexes：**

**Stability （稳定性）**

**Steady-state characteristics（稳态特性）**

**Dynamic characteristics （动态特性）**

## 3.4 Control System Stability Analysis



## 1. Concepts and Definitions

Stability of system is a prerequisite of regular operation, and is an important subject of control theory.

## (1). Concept of Stability

- A linear time-invariant system is called stable if it returns to its original equilibrium position when the disturbance effects disappears. By the contrary, such a system is unstable.
- We note that stability depends on zero input response of the system.

- **2. The Necessary and Sufficient Conditions for Stability**
- The definition shows that the stability a linear system only depends on the inherent characteristics of this system, while it is irrelevant with external conditions.
- Consider a system with initial conditions setting to zero and with an ideal unit pulse input $\delta(t)$ (***R(S)=1***).

  when t > 0, we have $\delta(t)$ =0. This is equivalent to a system whose output deviates from original equilibrium position.

  If the impulse response of the system meets the requirement:

$$\lim_{t \to \infty} c(t) = 0$$

  It means that the output converges to the original equilibrium position, and the system is stable

- **Consider a closed-loop system with transfer function to be:**

$$\Phi(s)=\frac{C(s)}{R(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{B(s)}{D(s)} \qquad (m\le n)$$

- Set the roots of the system characteristic equation $D(s)=0$ to be $p_i$ $(i=1,2,\cdots,n)$ and assume they are unequal with each other.
- The output is then:

$$C(s)=\frac{B(s)}{D(s)}R(s)=\frac{B(s)}{D(s)} \qquad k+2r=n$$

$$=\sum_{i=1}^{k}\frac{c_i}{s-p_i}+\sum_{j=1}^{r}\frac{\alpha_j s+\beta_j}{[s-(\sigma_j+j\omega_j)][s-(\sigma_j-j\omega_j)]}$$

- By using Laplace inverse transforms we have the output under ideal unit impulse:

$$c(t)=\sum_{i=1}^{k}c_i e^{p_i t}+\sum_{j=1}^{r}e^{\sigma_j t}(A_j\cos\omega_j t+B_j\sin\omega_j t) \qquad (t\ge 0)$$

- It shows that a necessary and sufficient condition for a linear system to be stable is that all the poles of the system transfer function have negative real parts. That is, a system is stable if all the poles of the transfer function are in the left-hand plane.

## 3. The Routh-Hurwitz Stability Criterion

- The system stability can be determined by the distribution of characteristic roots, while the root is determined by the equation coefficients. The Routh-Hurwitz stability method provides a answer to the question of stability by considering the characteristic equation of the system.
- The characteristic equation in the Laplace variable is written as:

$$a_n s^n + a_{n-1} S^{n-1} + \cdots\cdots + a_1 s + a_0 = 0$$

- we note that al the coefficients of the left polynomial must have the same sign (positive, for example). Also, it is necessary that all the coefficients be nonzero. These requirements are necessary but not sufficient.

## Steps of Routh-Hurwitz criterion:

- **Step 1:** Order the coefficients of the characteristic equation into an array or schedule as follows:

$$\begin{array}{l} a_n, a_{n-2}, a_{n-4} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n-1}, a_{n-3}, a_{n-5} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

- **Step 2:** Calculate corresponding elements and establish the Routh table (or Routh array).
- Consider a fifth-order system, the characteristic equation is:

$$a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

- the Routh-Hurwitz criterion table is

$$\begin{array}{llll}
s^5 & a_5 & a_3 & a_1 \\
s^4 & a_4 & a_2 & a_0 \\
s^3 & A_1 = \dfrac{a_4 a_3 - a_5 a_2}{a_4} & A_2 = \dfrac{a_4 a_1 - a_5 a_0}{a_4} & 0 \\
s^2 & B_1 = \dfrac{A_1 a_2 - a_4 A_2}{A_1} & B_2 = \dfrac{A_1 a_0 - 0}{A_1} = a_0 & 0 \\
s^1 & C_1 = \dfrac{B_1 A_2 - A_1 B_2}{B_1} & 0 & 0 \\
s^0 & D_1 = \dfrac{C_1 B_2 - 0}{C_1} = B_2 & 0 & 0
\end{array}$$

## Step 3: Determine the system stability by the Routh-Hurwitz criterion.

- The Routh-Hurwitz criterion states that: (It is a necessary and sufficient condition)
- **The system is stable if all values of the first column of the Routh array are positive and it's unstable if there is any negative value in the first column.**
- **The number of roots of q(s) with positive real parts is equal to the number of changes in sign of the first column of the Routh array.**
- **EX: The system characteristic equation is:**

$$s^4 + 7s^3 + 17s^2 + 17s + 6 = 0$$

- Solution: the Routh array is:

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 17 | 6 |
| $s^3$ | 7 | 17 | 0 |
| $s^2$ | 14.57 | 6 | 0 |
| $s^1$ | 14.12 | 0 | 0 |
| $s^0$ | 6 | 0 | 0 |

There is no change in sign of the first column, which means no roots of characteristic equation has a real part.

Thus, the system is stable.

- **EX: The system characteristic equation is:**

$$s^3 + 4s^2 + 10s + 50 = 0$$

- Determine the system stability by Routh-Hurwitz criterion.

- Solution: the Routh array is:

$$\begin{array}{lcc} s^3 & 1 & 10 \\ s^2 & 4 & 50 \\ s^1 & -2.5 & 0 \\ s^0 & 50 & 0 \end{array}$$

- Because two changes in sign appears in the first column, we find that two roots of the characteristic equation lie in the right-hand plane, thus the system is unstable.

- **2. Two Special Cases of Routh-Hurwitz criterion**
- (1) There is a zero in the first column, but some other elements of the row containing the zero in the first column are nonzero.
- If only one element in the array is zero, it may be replaced with a small positive number, $\varepsilon$, and we can complete the array element calculations just as before.
- **EX: the characteristic equation is** $s^5+s^4+5s^3+5s^2+2s+1=0$
- Determine the system stability.
- Solution: the Routh array is:

| | | | |
|---|---|---|---|
| $s^5$ | $1$ | $5$ | $2$ |
| $s^4$ | $1$ | $5$ | $1$ |
| $s^3$ | $0(\varepsilon)$ | $1$ | $0$ |
| $s^2$ | $\dfrac{5\varepsilon-1}{\varepsilon}$ | $1$ | $0$ |
| $s^1$ | $\dfrac{5\varepsilon-1-\varepsilon^2}{5\varepsilon-1}$ | $0$ | $0$ |
| $s^0$ | $1$ | $0$ | $0$ |

- Because two changes in sign appears in the first column, we find that two roots of the characteristic equation lie in the right-hand plane, thus the system is unstable.

- **Case 2. There is a zero in the first column, and the other elements of the row containing the zero are also zero.**
- This occurs when the characteristic equation has conjugate complex roots or conjugate imaginary roots.
- This problem is circumvented by utilizing the auxiliary polynomial. The all zero row can be replaced by the equation confidents obtained according to the derivation of the auxiliary polynomial.
- **EX: the characteristic equation is** $s^3 + s^2 + 16s + 16 = 0$

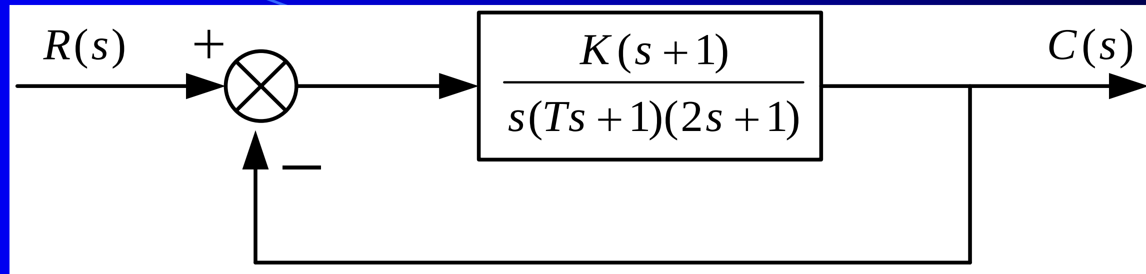
Determine the system stability.

- Solution: the Routh array is:

$$\begin{array}{lcc} s^3 & 1 & 16 \\ s^2 & 1 & 16 \\ s^1 & 0(2) & 0 \\ s^0 & 16 & 0 \end{array}$$

- Establish a auxiliary equation with the elements above the all zero row. $A(s) = s^2 + 16 = 0$ $s_{1,2} = \pm 4j$
- we get a pair of conjugate imaginary roots, thus the system is critical stable.

- **Determine K, T to make the system stable**




- Solution:
- The system characteristic equation is:

$$2Ts^3 + (2+T)s^2 + (K+1)s + K = 0$$

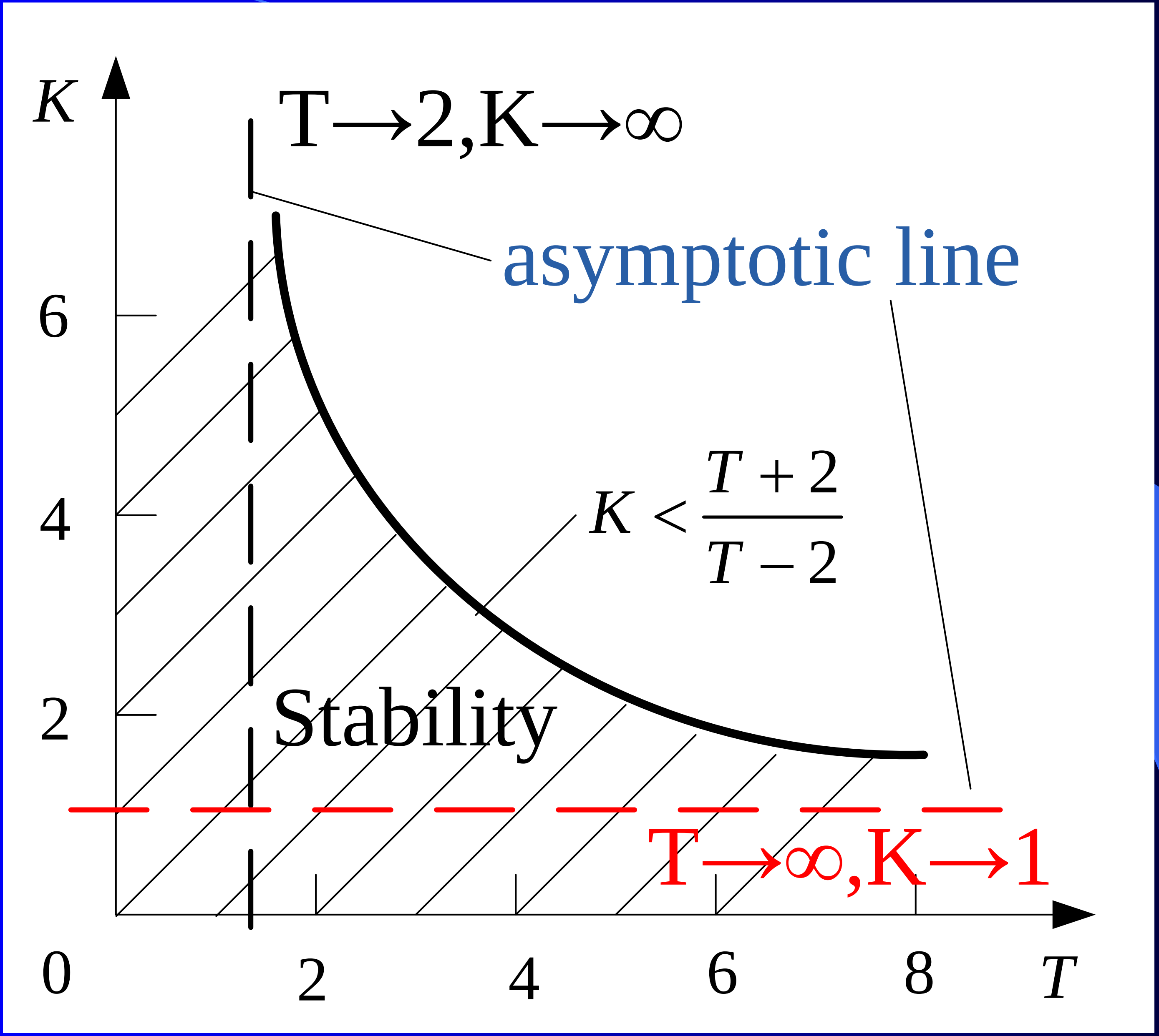
- the Routh array is:

$$\begin{array}{lcc} s^3 & 2T & K+1 \\ s^2 & 2+T & K \\ s^1 & \dfrac{(2+T)(K+1)-2TK}{2+T} & 0 \\ s^0 & K & 0 \end{array}$$

- To make the system stable, the first column element signs should be all positive. Thus we have:

$$K > 0, 2T > 0, \qquad (2+T)(K+1) - 2TK > 0$$

- And the value ranges are: $T > 0$, $\quad 0 < K < \dfrac{T+2}{T-2}$



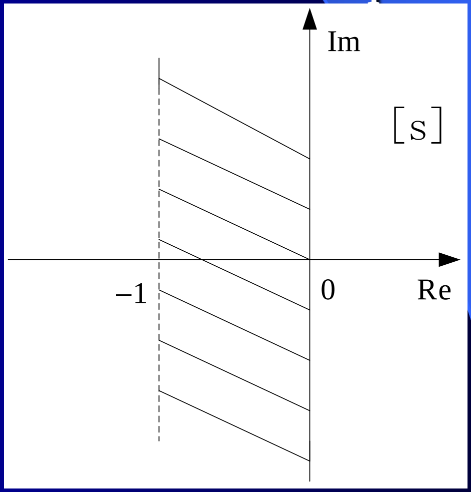
K
T→2,K→∞
asymptotic line
6
4
2
$K < \frac{T+2}{T-2}$
Stability
T→∞,K→1
0
2
4
6
8
T

- **EX. the characteristic equation is** $s^3+8s^2+10s+2=0$
- **Determine the system stability and the number of the roots between the plumb-line** $s=-1$ **and the imaginary axis.**
- Solution: the Routh array is:

$$\begin{array}{lcc} s^3 & 1 & 10 \\ s^2 & 8 & 2 \\ s^1 & 9.75 & 0 \\ s^0 & 2 & 0 \end{array}$$

- There is no change in sign of the first column, thus the system is stable.
- Set $s=s_1-1$ and substitute into the original characteristic equation, we have: $s_1^3+5s_1^2-3s_1-1=0$

$$\begin{array}{lcc} s_1^3 & 1 & -3 \\ s_1^2 & 5 & -1 \\ s_1^1 & -2.8 & 0 \\ s_1^0 & -1 & 0 \end{array}$$




- There is one change in sign of the first column, thus there is one root lies between the plumb-line $s=-1$ and the imaginary axis.

# Exercise in Classroom

- Use Routh's stability criterion to determine the range K for system is stable.

$$S^4 + 22S^3 + 10S^2 + 2S + K = 0$$

# Homework

- Page 124
  - 3.38
  - 3.39
  - 3.40 ( Matlab Tool )
- Deadline 15.OCT.2012

GOOD HOLIDAY!

# Chapter 3 Dynamic Response

## -3.5 Control System Steady-state characteristics

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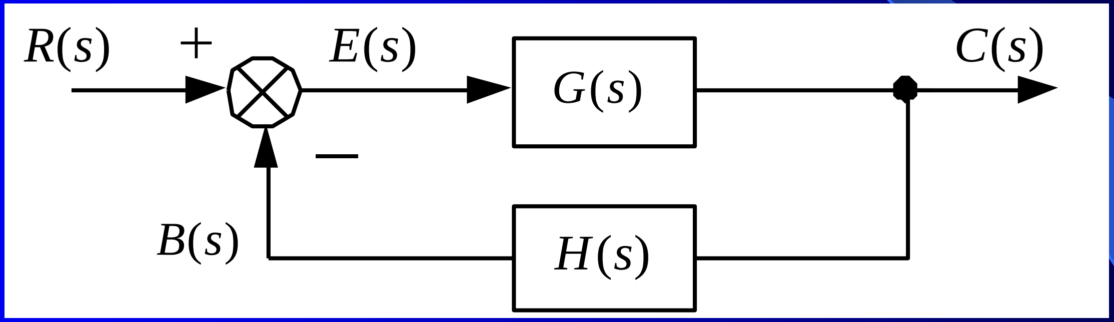
# 3.5 Steady-state Analysis in Time Domain

- **Three performance indexes：**
  - Stability
  - Steady-state characteristics
  - Dynamic characteristics
- For a stable control system, Steady-state error (稳态误差)is a measurement of control accuracy. Steady-state error is also known as the steady-state performance.
- Researches show that steady-state error is relevant to system structure and input signal. One of the task to design a control system is to minimize or even eliminate the steady-state error on the prerequisite of system stability.

# 3.5.1 The Basic Concepts of Error

**Error and Steady-state Error**

- **(1) Definition**
- two kinds of definitions of error:




- a. Definition in the input port: the difference of actual value and expected value(真值&理论值) of the system output.
- This method is often used in performance indexes analysis. However, it's immesurable in some occasions, thus it only has mathematical meanings.
- b. Definition in the output port: the difference of input signal and main feedback signal.

$$e(t) = r(t) - b(t)$$

or $E(s)=R(s)-B(s)=R(s)-C(s)H(s)=R(s)[1+G(s)H(s)]$

Consider $\Phi_e(s)=\dfrac{E(s)}{R(s)}=\dfrac{1}{1+G(s)H(s)}$

and $\Phi_e(s)$ ——Error transfer function of the system

we have: $E(s)=\Phi_e(s)R(s)$

Error of this definition is mesurable if actual system, thus it has phisical meaning. We use definition in the input port to analyze and caculate system error. Since error is a function of time, we have the expression in time domain:

$$e(t)=L^{-1}[E(s)]=L^{-1}[\Phi_e(s)R(s)]=e_{ts}(t)+e_{ss}(t)$$

and: $e_{ts}(t)$ —— Dynamic component（动态分量）

$e_{ss}(t)$ —— Steady-state component （稳态分量）

- **(2) Steady-state error** $e_{ss}$ : the steady-state component $e_{ss}(\infty)$ of the error signal $e_{ss}(t)$ .
- For a stable system, the system dynamic process comes to an end as time goes to infinity and $e_{ts}(t)$ will tend to zero. According to Laplace final value theorem, the steady-state error of a stable non-unit feedback system is:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{R(s)}{1+G(s)H(s)}$$

- From the equation above, we know that the steady-state error is relevant to the input signal and the open-loop transfer function structure.
- The steady-state error is determined by the system structure described by open-loop transfer function once the input signal form is fixed.

# How to understand the steady-state error is determined by the system structure?

- The open-loop transfer function $G(s)H(s)$ can defined with the Zeros/ Poles expression:

$$G(s)H(s) = \frac{K_r \prod_{i=1}^{m} (s \underset{-z_i}{+} z_i)}{s^v \prod_{j=1}^{n} (s + p_j)}$$

Where $-z_i$ and $-p_j$ are the zeros/ poles of open-loop transfer function , $K_r$ is the amplify coefficient.

# How to understand the steady-state error is determined by the system structure?

- Let $K = \dfrac{K_r \prod_{i=1}^{m} z_i}{\prod_{j=1}^{n-v} p_j}$

then the steady-state error of a stable non-unit feedback system is:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{R(s)}{1+G(s)H(s)}$$

$$= \lim_{s\to 0} sR(s)\frac{1}{1+\dfrac{K_r \prod_{i=1}^{m}(s+z_i)}{s^v \prod_{j=v+1}^{n}(s+p_j)}} = \lim_{s\to 0} sR(s)\frac{1}{1+\frac{K}{s^v}}$$

## How to understand the steady-state error is determined by the system structure?

- So the poles number of open loop transfer function $v$ ; the amplify coefficient $K$ ; and the input signal $R(s)$ are determine the steady-state error $e_{ss}$
- We select three typical input signals to analysis the $e_{ss}$
  - 1） **Step Input** $R(s)=\frac{1}{S}$
  - 2） **Ramp Input** $R(s)=\frac{1}{s^2}$
  - 3） **Acceleration Input** $R(s)=\frac{1}{S^3}$

# The Type for Control system with poles number

- The open-loop transfer function $G(s)H(s)$ is defined with the Zeros/ Poles expression:

$$G(s)H(s)=\frac{K_r\prod_{i=1}^{m}(s+z_i)}{s^v\prod_{j=1}^{n}(s+p_j)}$$

When: $v=0$, it is called as type-0 system;

$v=1$, it is called as type-1 system;

$v=2$, it is called as type-2 system

**EX: The open-loop transfer function is**

$$G(s)H(s)=\frac{20}{(0.5s+1)(0.04s+1)}$$

**Determine the steady-state error** $e_{ss}$ **when**

**r(t)=1(t), r(t)=t**

Solution:

$$e_{ss}=\lim_{s\to 0} s\frac{1}{1+G(s)H(s)}R(s)=\lim_{s\to 0} s\frac{(0.5s+1)(0.04s+1)}{(0.5s+1)(0.04s+1)+20}R(s)$$

Unit Pulse function r(t) = 1(t), we have R(s)=1/s

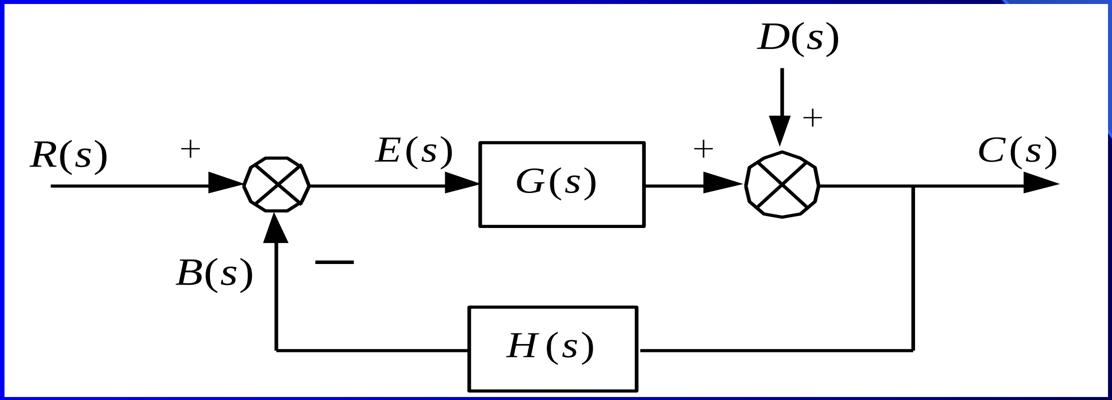
$$e_{ss}=\lim_{s\to 0} s\frac{(0.5s+1)(0.04s+1)}{(0.5s+1)(0.04s+1)+20}\bullet\frac{1}{s}=\frac{1}{21}\approx 0.05$$

r(t) = t, we have R(s)=1/s$^2$

$$e_{ss}=\lim_{s\to 0} s\frac{(0.5s+1)(0.04s+1)}{(0.5s+1)(0.04s+1)+20}\bullet\frac{1}{s^2}=\infty$$

- **2. Steady-state error under disturbance signal**

  Systems are often suffering a variety of disturbances. Such as: load torque change, voltage and frequency fluctuation, temperature change. Therefore, the steady-state error under disturbance signals represent the anti-disturbance capability of the system.




- The Laplace transform expression of the output signal is:

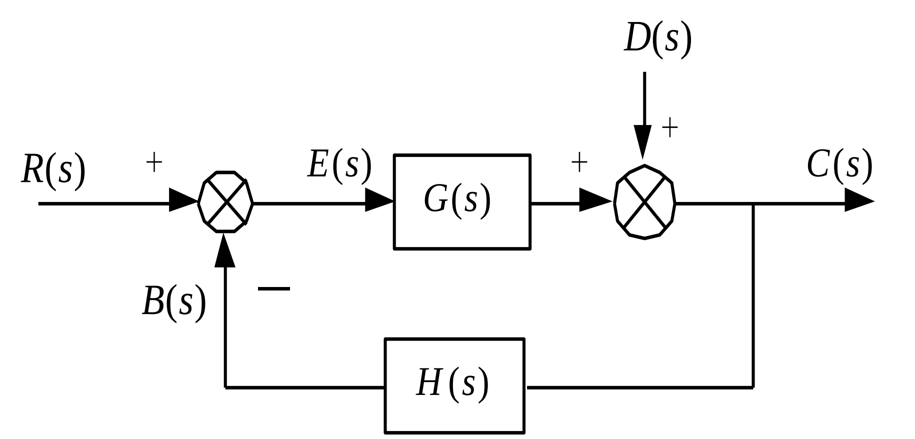
$$C(s) = D(s) + E(s)G(s) = D(s) + G(s)\left[R(s) - H(s)C(s)\right]$$

$$C(s) = \frac{D(s)}{1+G(s)H(s)} + \frac{G(s)}{1+G(s)H(s)} \cdot R(s)$$

$R(s)=0$ : $C(s)=\dfrac{D(s)}{1+G(s)H(s)}$

$E(s) = -H(s)C(s)$

$$=\frac{-H(s)}{1+G(s)H(s)}\cdot D(s)$$




We take the absolute value as the error when system reaches steady-state.

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0}\frac{H(s)}{1+G(s)H(s)}\cdot sD(s)$$

Assume that the disturbance is a step signal, that is $D(s)=\dfrac{1}{s}$ ,we have:

$$e_{ss}=\frac{H(0)}{1+G(0)H(0)}\approx\frac{1}{G(0)}$$ $(G(0)H(0)>>1)$

and: $G(0)=\lim\limits_{s\to 0} G(s)$

From the analysis above we note that:

The steady-state error caused by the disturbance decreases as the forward path coefficient in front of disturbance node increases.

Therefore, in order to reduce the steady-state error caused by the disturbance, we can increase the forward path coefficient in front of disturbance node, or we can insert a integral element in front of disturbance node. However, these will decrease the system stability.

# 3.5.2 Steady-state Error Coefficients

## 1. Steady-state error under different signals

### （1）Step Input(阶跃信号)

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0} s\cdot\frac{1}{1+G(s)H(s)}\cdot\frac{1}{s}=\frac{1}{1+\lim\limits_{s\to 0}G(s)H(s)}$$

Definition： $K_p=\lim\limits_{s\to 0}G(s)H(s)$ ----position error constant。

As for a type-0 system: $v=0$

$$K_p=\lim_{s\to 0}\frac{K(1+\tau_1 s)(1+\tau_2 s)\cdots\cdots(1+\tau_m s)}{(1+T_1 s)(1+T_2 s)\cdots\cdots(1+T_n s)}=K \qquad (N=0)$$

As for a type-1 or higher system: $v=1$

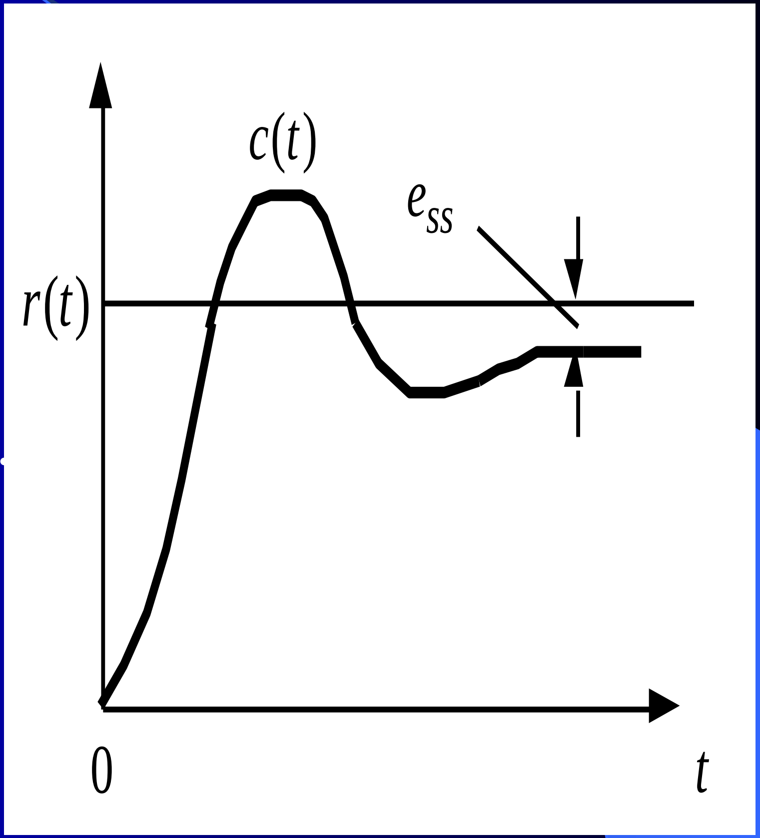
$$K_p=\lim_{s\to 0}\frac{K(1+\tau_1 s)(1+\tau_2 s)\cdots(1+\tau_m s)}{s^N(1+T_1 s)(1+T_2 s)\cdots(1+T_{n-N} s)}=\infty \qquad (N\geq 1)$$

- Therefore, the steady-state error can be represented as:

$$e_{ss}=\begin{cases}\dfrac{1}{1+K_p} & K_p=K \quad N=0\\ 0 & K_p=\infty \quad N\geq 1\end{cases}$$

- the steady-state error of type-0 system for a step input is a constant. The magnitude of $e_{ss}$ is inversely proportional to the open-loop amplification factor $K$ . $e_{ss}$ decreases as $K$ increases. However, the error will not go to zero unless $K$ goes to infinite. Thus the type-0 system is also called the discrepancy system. To reduce the steady-state error $e_{ss}$, we can increase the open-loop amplification factor $K$ on the prerequisite of system stability.

（please see page 72, Fig3-12（a））



## (2) Ramp Input(斜坡/速度输入)

r(t)=t therefore $R(s)=\frac{1}{s^2}$ steady-state error is

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0} s\cdot\frac{1}{1+G(s)H(s)}\cdot\frac{1}{s^2}=\lim_{s\to 0}\frac{1}{sG(s)H(s)}$$

Definition: $K_v=\lim_{s\to 0} s\cdot G(s)H(s)$ **--velocity error constant**

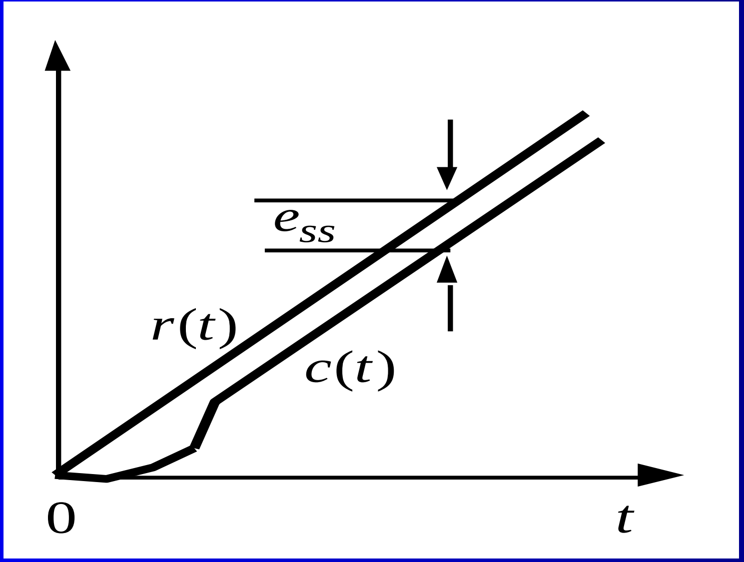
type-0 system: $K_v=0$

type-1 system : $K_v=K$

type-2 or higher system: $K_v=\infty$

- $$e_{ss} = \begin{cases} \infty & K_v = 0 \quad N = 0 \\ \dfrac{1}{K_v} & K_v = K \quad N = 1 \\ 0 & K_v = \infty \quad N \geq 2 \end{cases}$$

The output of type-1 system can track the velocity input, but a error always exits. Therefore, $K_v$ ( $K$ ) must have enough magnitude to constrain the error as expected.




（ please see page 72, Fig3-12（b））

## (3) Acceleration Input(加速度输入)

r(t)=t$^2$ /2 therefore: $R(s)=\frac{1}{s^3}$

steady-state error is:

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0} s\cdot\frac{1}{1+G(s)H(s)}\cdot\frac{1}{s^3}=\lim_{s\to 0}\frac{1}{s^2G(s)H(s)}$$

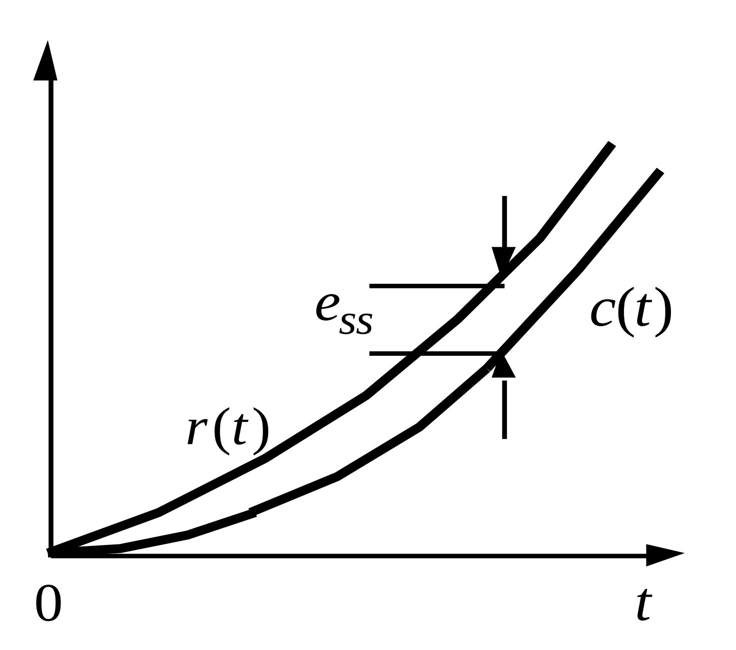
definition: $K_a=\lim_{s\to 0} s^2\cdot G(s)H(s)$ --acceleration error constant

type-0 system: $K_a=0$

type-1 system: $K_a=0$

type-2 system: $K_a=K$

type-3 or higher system: $K_a=\infty$




（please see page 72, Fig3-12（c））

•

$$e_{ss} = \begin{cases} \infty & K_a = 0 \quad N < 2 \\ \dfrac{1}{K_a} & K_a = K \quad N = 2 \\ 0 & K_a = \infty \quad N \geq 3 \end{cases}$$

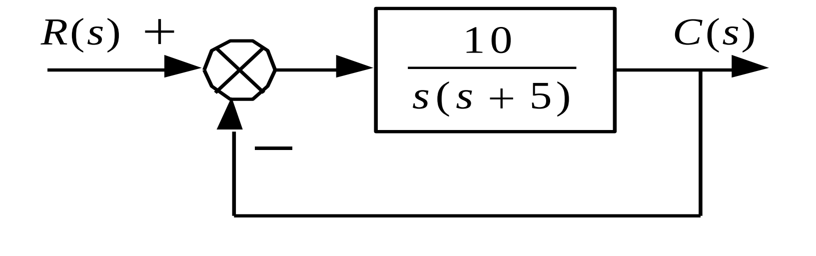
The steady-state error of a type-2 system is a constant when there is an unit acceleration input.

# Table Summary of steady-state error

| Type Number | Error constants Kp Kv Ka | Unit-step input $r(t)=u(t)$ | Unit-ramp input $r(t)=t$ | Unit-acceleration input $r(t)=\frac{1}{2}t^2$ |
|---|---|---|---|---|
| 0 | $K$ $0$ $0$ | $\frac{1}{1+K}$ | $\infty$ | $\infty$ |
| $I$ | $\infty$ $K$ $0$ | 0 | $\frac{1}{K}$ | $\infty$ |
| $II$ | $\infty$ $\infty$ $K$ | 0 | 0 | $\frac{1}{K}$ |

- 1. Steady-state error is relevant to input signal and system structure.
- 2. Ways to reduce or eliminate the steady-state error:
- a. increase the open-loop amplification factor K;
- b. increase the type number of G(s).

**EX:** Consider the following system， when the system input is r(t) = $u(t)$, $t$ and $\frac{1}{2}t^2$ determine the corresponding steady-state errors.




Solution： This is a type-1 system, therefore:

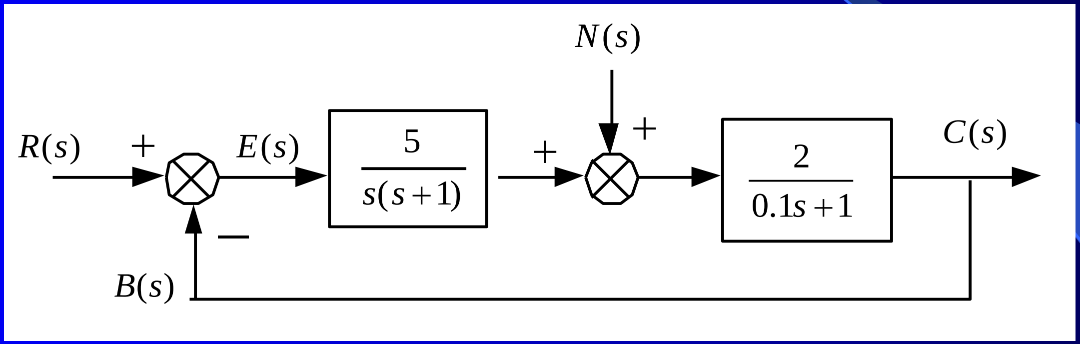
$$K_p = \infty \qquad K_v = 2 \qquad K_a = 0$$

And the steady-state errors are:

$$e_{ssP} = \frac{1}{1+K_P} = 0 \qquad e_{ssv} = \frac{1}{K_v} = \frac{1}{2} \qquad e_{ssa} = \frac{1}{K_a} = \infty$$

- While the errors determined by the steady-state error coefficients would be zeros, constants or infinities, these values do not reflect the regularity of error changing with time.
- Therefore, the **dynamic error coefficient** was introduced in some books.

- EX： Consider the following system， when the system input is r(t)=t and n(t)= －1(t), determine the corresponding steady-state errors.




Solution：

(1) The effect of control signal （Set N(s)=0）

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{s(0.1s+1)(s+1)}{s(0.1s+1)(s+1)+10}$$

$$e_{ssn} = \lim_{s\to 0} s\frac{s(0.1s+1)(s+1)}{s(0.1s+1)(s+1)+10} \bullet \frac{1}{s^2} = 0.1$$

(2) The effect of disturbance signal（Set R(s)＝0）

$$\frac{E(s)}{N(s)} = \frac{-\dfrac{5}{s(s+1)}}{1+\dfrac{5}{s(s+1)} \bullet \dfrac{2}{0.1s+1}}$$

$$e_{ssn} = \lim_{s\to 0} s \frac{-\dfrac{5}{s(s+1)}}{1+\dfrac{5}{s(s+1)} \bullet \dfrac{2}{0.1s+1}} \bullet \frac{-1}{s} = \frac{5}{10} = 0.5$$

Overall system error is：

$$e_{ss} = e_{ssr} + e_{ssn} = 0.1 + 0.5 = 0.6$$

# Exercise in Classroom:

- Open loop transfer function:

(1) $$G(s)H(s)=\frac{10}{s(s+1)(s+2)}$$

- **Stability?**

(2) $$G(s)H(s)=\frac{3(2s+1)}{s(s-1)(s+2)}$$

- **Stability?**

Calculate the **Steady-state coefficients:**

$K_p$ $K_v$ $K_a$

# Chapter 3 Dynamic Response

## -3.6 Control System Dynamic characteristics

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# 3.6 Dynamic Analysis in Time Domain

**Three performance indexes:**

- **Stability**
- **Steady-state characteristics**
- **Dynamic characteristics**

System output:

$$\mathbf{c(t) = c_t(t) + c_s(t)}$$

$c_t(t)$ - dynamic component (or transient component)

$c_s(t)$ - steady-state component

- Dynamic response of the control system (or transient response) refers to the system response from the initial state to the steady-state.
- Input signal only affects the steady-state component.
- The accuracy of the system analysis depends on the authenticity of mathematical model.
- Dynamic response analysis is based on the system stability.
- Dynamic response of unstable system is divergent.

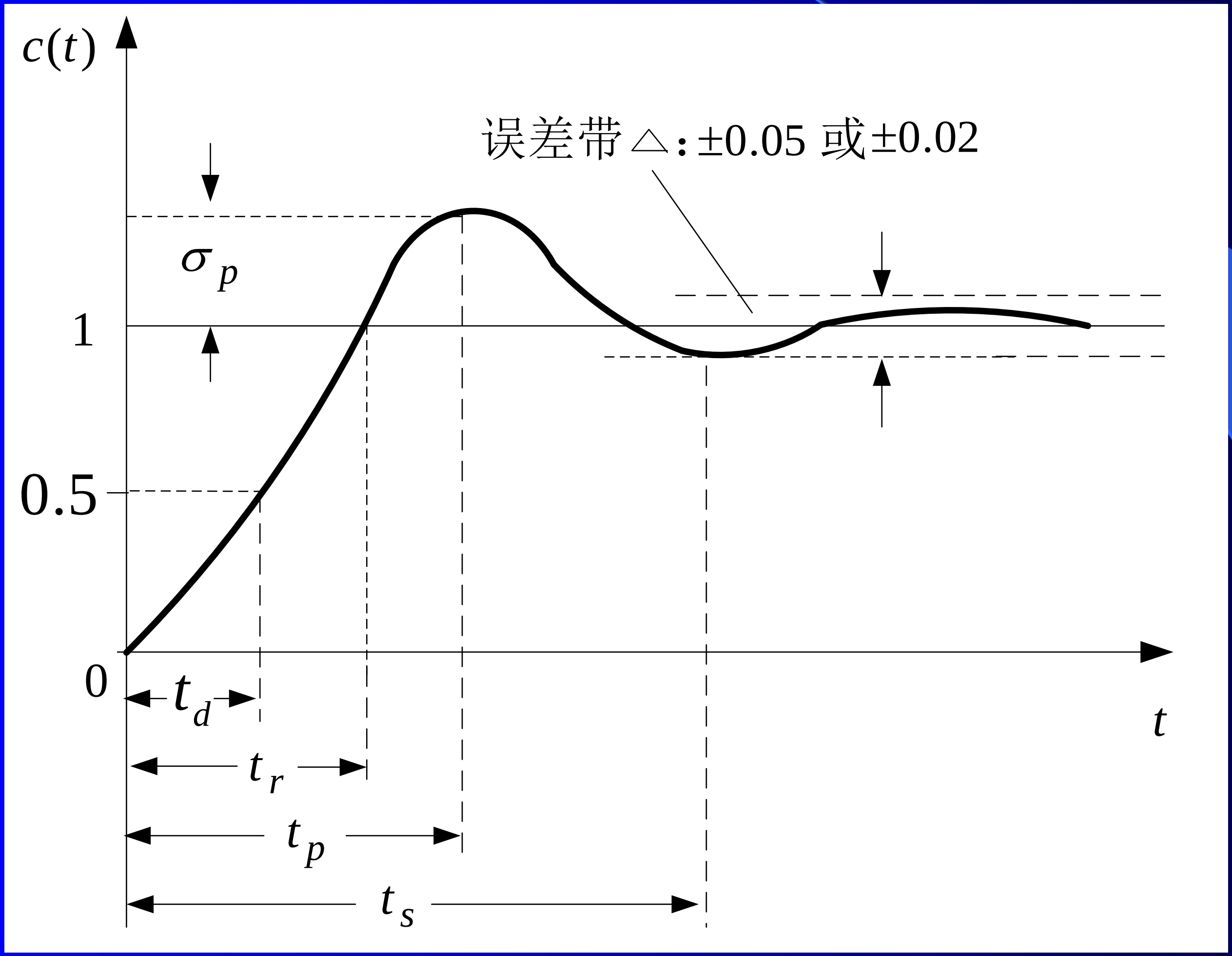
- **1. Dynamic Performance Index**
- The step input signal is the easiest to generate and evaluate and is usually chosen for time-domain performance tests.
- **1. overshoot (Mp 超调量)**: the amount by which the system output response proceeds beyond the desired response.

  In the following formula, $c(t_P)$ is the peak value of the time response, and $c(\infty)$ is the final value of the response.

$$\sigma_P = \frac{c(t_P) - c(\infty)}{c(\infty)} \times 100\%$$

- **2. Delay time（td 延迟时间）**: half of the time for the response reaching the final value for the first time.
- **3. Peak Time（tp 峰值时间）:** the time for a system to respond to a step input and rise to a peak response.
- **4. Rise time（tr 上升时间）:** the time for the dynamic response rising from zero to the steady-state value for the first time (choose 10-90% of the steady-state value, if there is no overshoot).

5. **Setting time（ts 调整时间）** (or transition process time): the time required for the system to settle within a certain percentage, △, of the input amplitude (or error band). This band can be set as ±2% or ±5%.

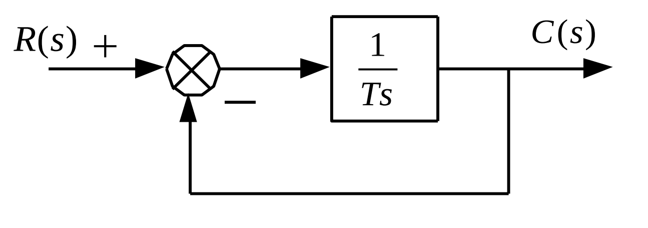


## 3.6.1 Performance of a first-order system

- We expect the system to have a swift response. That is, the system output can change with the control signal swiftly.

- **1 Introduction**

$$\frac{C(s)}{R(s)} = \frac{1}{1+Ts} \qquad C(s) = \frac{1}{s(Ts+1)} = \frac{1}{s} - \frac{1}{1+Ts}$$




- **Unit-step response**

$$c(t) = 1 - e^{-\frac{t}{T}} \qquad (t > 0)$$

- **Features:**
- (1) when t = T, the output reaches 0.632 of the magnitude of $e_{ss}(\infty)$

$$c(t) = 1 - e^{-\frac{t}{T}}$$

- $c(0) = 1 - e^{-0} \approx 0$ - T = 0, the output is 0
- $c(T) = 1 - e^{-1} \approx 0.632$ - T = ∞, the output reaches the steady-state value
- $c(3T) = 1 - e^{-3} \approx 0.95$ - T = T, the output reaches 0.632 of the magnitude of $e_{ss}(\infty)$ ;
- $c(4T) = 1 - e^{-4} \approx 0.98$ - T = 3T, the output reaches 0.95 of the magnitude of $e_{ss}(\infty)$ ;
- $c(\infty) = 1 - e^{-\infty} \approx 1$ - T = 4T, the output reaches 0.98 of the magnitude of $e_{ss}(\infty)$ ;
- (2) when t = 0, the tangent slope of the response curve is 1/T, the intersection of tangent with the steady-state value. the tangent slope of c(t) declines as ***t*** increases

$$\frac{dc(t)}{dt} = \frac{1}{T} e^{-\frac{t}{T}} \qquad \left.\frac{dc(t)}{dt}\right|_{t=0} = \frac{1}{T} \qquad \left.\frac{dc(t)}{dt}\right|_{t=T} = 0.368\frac{1}{T} \qquad \lim_{t\to\infty}\frac{dc(t)}{dt} = 0$$

- **(3) Setting time:** **$t_s$=3T（95%）， $t_s$=4T（98%）**
- **(4) Delay time:** **$t_d$≈0.69T**

$$c(t_d) = 1 - e^{-\frac{t}{T}} = 0.5 \qquad t_d = 0.69T$$

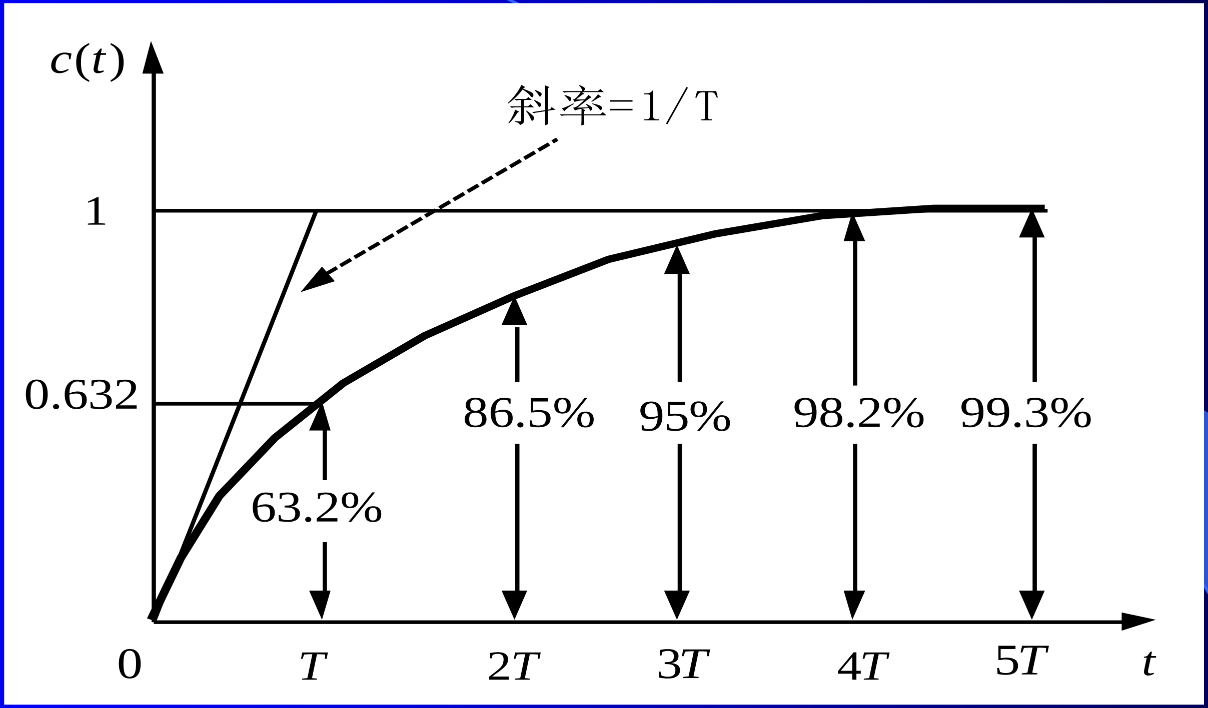
- **(5) Rising time:** **$t_r$≈0.22T**

$$c(t) = 1 - e^{-\frac{t}{T}} = 0.1 \qquad t = 0.1T$$

$$c(t) = 1 - e^{-\frac{t}{T}} = 0.9 \qquad t = 2.3T$$

- **∴ $t_r$=2.3T-0.1T=2.2T**
- **(6) Eigenvalue is *S=－1/T*， and system has better dynamic and steady-state performance as T declines.**

- the response of a first-order system

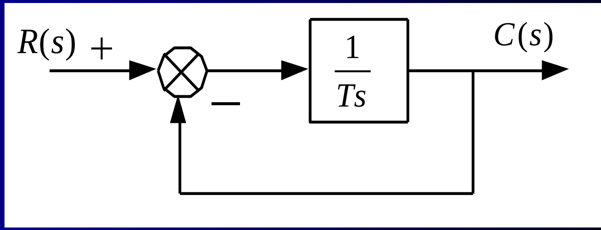


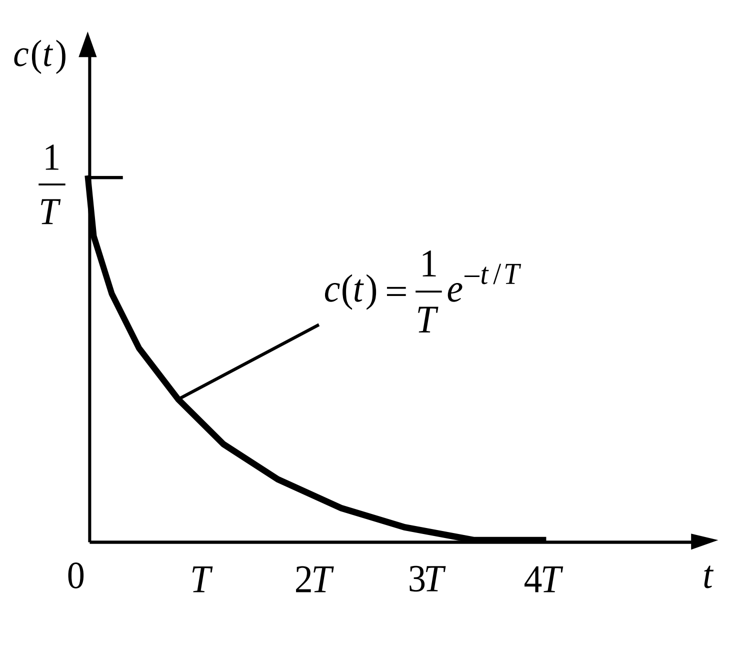

- This is an exponential curve and the slope reaches the maximum ***1/T*** when ***t=0***. If the response rises at such a speed, it would have reached the steady-state value at t = T, however, the output reaches 0.632 of the steady-state value by then in a practical system, and after 3T and 4T the output reaches 0.95 and 0.98 of $e_{ss}$.

# Unit impulse response

$$C(s) = \frac{1}{1+Ts} \bullet R(s) = \frac{1}{1+Ts}$$

$$g(t) = c(t) = \frac{1}{T} e^{-\frac{t}{T}}$$



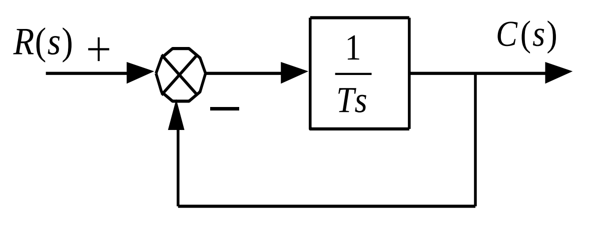




- Unit pulse response is also an exponential curve, and it is 1/T when t = 0.
- We note that the unit impulse response is the derivative of unit step response, while the unit step response is the integral of unit impulse response.

## Unit ramp response

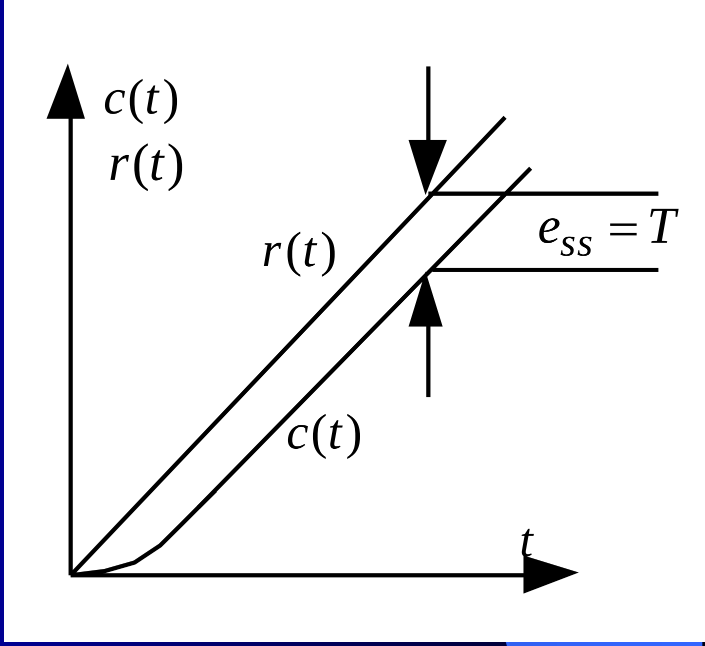
$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{1}{1+Ts} \bullet \frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+1/T}$$




output:

$$c(t) = t - T - Te^{-t/T}$$




The response consists of two parts:

the steady-state component is ***(t-T)*,**

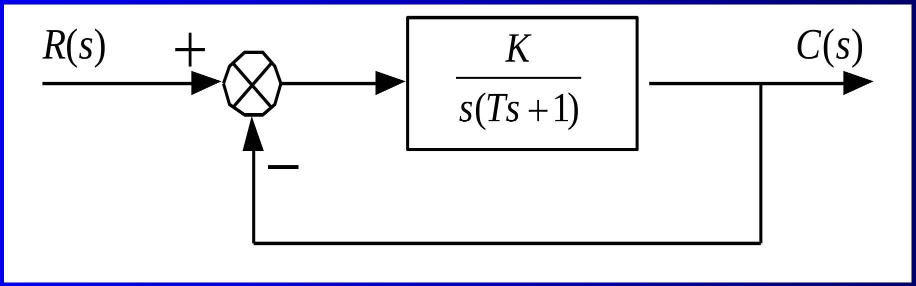
and it is also a unit ramp,

but there is a delay time of ***T***,

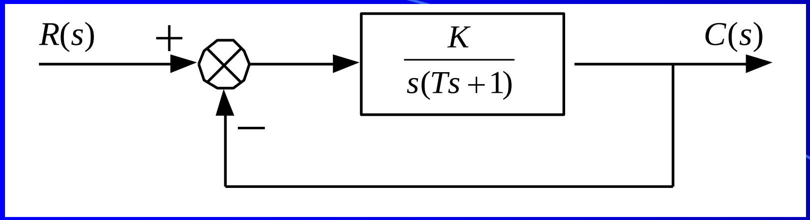
also it is the steady-state error;

the transient component is ***$Te^{-t/T}$***, and it attenuates to zero with the attenuation rate of ***1/T***. The steady-state error declines as T declines.

- **3.6.2 Transient Response of second-order system**
- **Systems that can be described by second-order differential equations are known as the second-order system. In physical, a second-order system contains two separate storage elements, such as inductor and capacitor.**
- **1. standard form of the second-order system**




$$\Phi(s)=\frac{C(s)}{R(s)}=\frac{\frac{K}{T}}{s^2+\frac{1}{T}s+\frac{K}{T}}$$




$$\Phi(s) = \frac{C(s)}{R(s)} = \frac{\frac{K}{T}}{s^2 + \frac{1}{T}s + \frac{K}{T}}$$

set: $\omega_n^2 = \dfrac{K}{T}$ $\quad 2\zeta\omega_n = \dfrac{1}{T}$ $\quad$ and the standard form is:

$$\Phi(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n$ ——undamping natural frequency(无阻尼振荡角频率);

$\zeta$ ——damp radio （阻尼比）

- characteristic equation of the second-order system is:

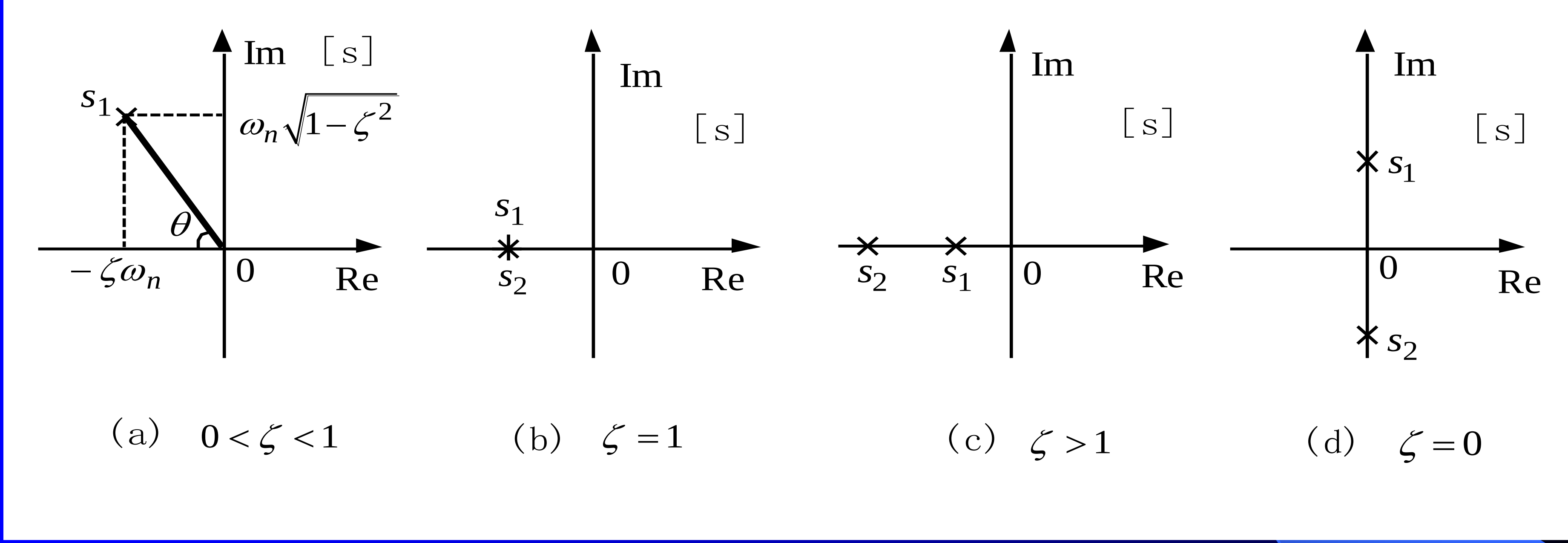
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad (0 < \zeta < 1)$$

- Two characteristic roots (closed-loop poles) are:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

The features of roots are relevant to $\zeta$ , we will discuss the following four cases.

the distribution of characteristic roots [s] in s-plane: $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$




- **1. Underdamping system:** $(0<\zeta<1)$
- $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ $\quad s_{1,2} = -\sigma \pm j\omega_d$ $\quad tg\,\theta = \sqrt{1-\varsigma^2}/\varsigma$
- Thus the second-order system has a pair of conjugate complex roots:
- Where: $\sigma = \zeta\omega_n$ - the attenuation coefficient
- $\omega_d = \omega_n\sqrt{1-\zeta^2}$ - the oscillation frequency

when the input is unit step, the Laplace transform of the output is:

$$C(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

apply the inverse Laplace transform to the equation above, we have the unit step response:

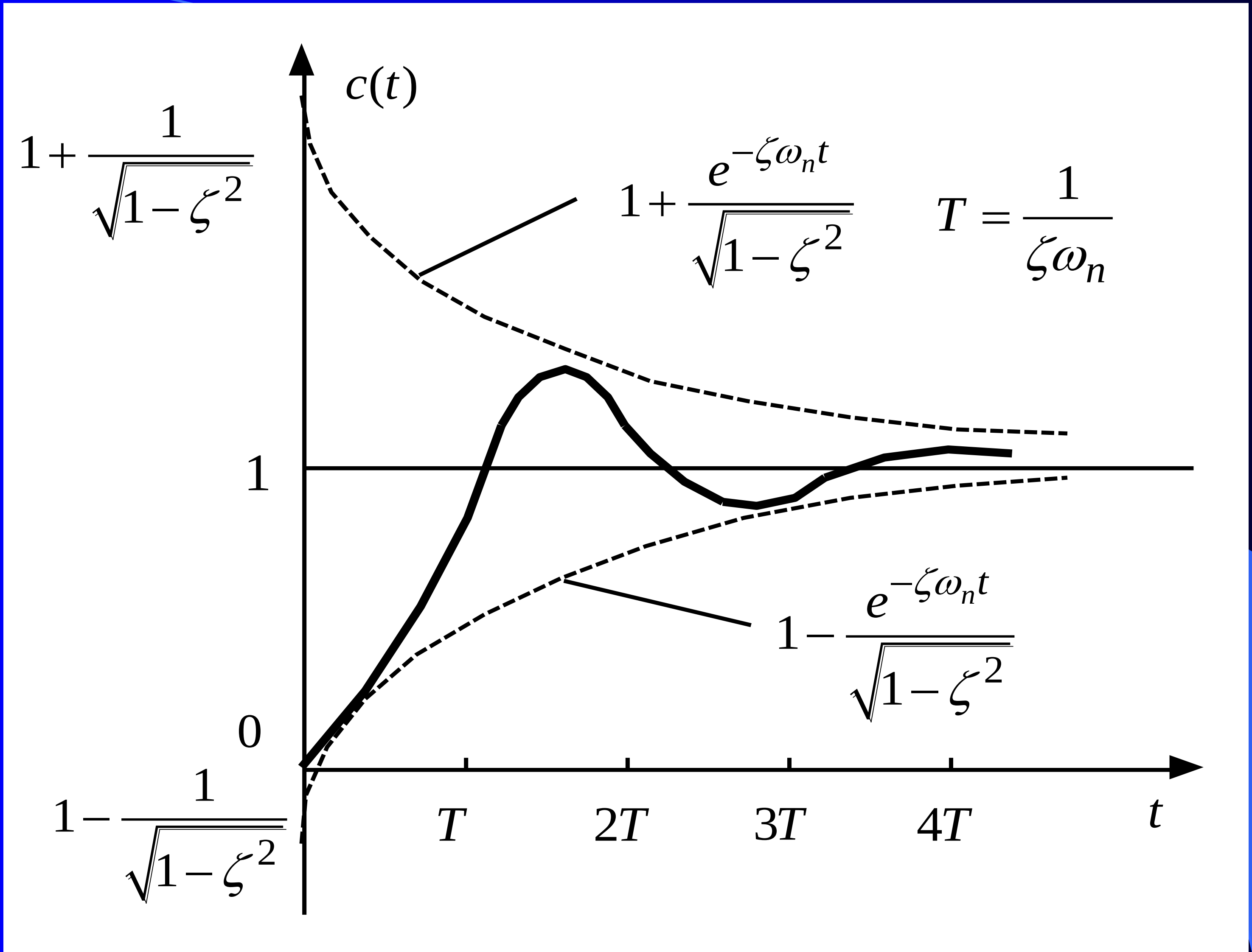
$$c(t)=1-e^{-\zeta\omega_n t}(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}(\sqrt{1-\zeta^2}\cos\omega_d t+\zeta\sin\omega_d t)$$

$$=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$$

$$\theta=\arccos\zeta$$

where：

- The step response of a underdamping second-order system is an oscillatory attenuating curve.
- Oscillatory frequency is $\omega_d$, and $1\pm\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$ is the envelope for the dynamic response. The time constant of the envelop is $\sigma=\zeta\omega_n$. The step response ***C(t)*** is always constrained by a pair of envelops and the convergence rate is determined by the value of time constant $\sigma=\zeta\omega_n$, thus $1/\varsigma\omega_n$ is also known as attenuation coefficient.




System oscillate more sharply as $\zeta$ declines. $\zeta$ is usually chosen as 0.5—0.8.

## 2. dynamic response performance indexes of second-order systems

(1) peak time $t_P$

since:

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p} = 0$$

$$\varsigma\omega_n e^{-\varsigma\omega_n t_p} \sin(\omega_d t_p + \theta) - \omega_d e^{-\varsigma\omega_n t_p} \cos(\omega_d t_p + \theta) = 0$$

we have:

$$tg(\omega_d t_p + \theta) = \frac{\sqrt{1-\varsigma^2}}{\varsigma}$$

$$\omega_p t_p = 0, \pi, 2\pi, 3\pi \cdots\cdots$$

- $t_P$ is the time that the response reaches the peak value for the first time. we choose that: $\omega_d t_P = \pi$
- Since: $tg\theta = \sqrt{1-\varsigma^2} / \varsigma$
- we have: $$t_P = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$t_P$ is inversely proportional to the imaginary part of the poles. Consider that $\zeta$ is a constant and we note that $t_P$ declines as the poles go farther away from the real axis.

## 3.6.2 Transient Response of second-order system

### (2) Overshot $\sigma_P$

The maximum overshot occurs at the peak time, thus we substitute

$$t_P = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$ into $$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

And we have:

$$c(t_P) = 1 - e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$

$$\sigma_P = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

It indicates that the overshot of a second-order system is relevant to the damping ratio only, $\sigma_P$ declines as $\zeta$ increases.

## (3) setting time $t_s$

The attenuation of the underdamping response can be represented by the envelop approximately.

$$|1-c(t_s)| \approx \left|1-\left(1\pm\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}}\right)\right| = 0.05$$

( $\triangle = 0.05$ )

We have $t_s = -\frac{1}{\zeta\omega_n}\left[\ln 0.05 + \ln\sqrt{1-\zeta^2}\right]$

When $\zeta << 1$ , ignore the factor $\ln\sqrt{1-\zeta^2}$

$$t_s = -\frac{-\ln 0.05}{\zeta\omega_n} \approx \frac{3}{\zeta\omega_n}$$

$\triangle = 0.05$

$$t_s \approx \frac{4}{\zeta\omega_n}$$

$\triangle = 0.02$

It indicates that the setting time is inversely proportional to the real part of poles. Since $\sigma_P$ is determined by $\zeta$ , if $\zeta$ is fixed, $t_s$ declines as $\omega_n$ increases. Thus we can accelerate system response speed without affecting system overshoot.

(4) Rise time $t_r$

According to the definition,

$$c(t_r) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 1$$

since: $e^{-\zeta\omega_n t_r} \neq 0$

We have: $\omega_d t_r + \theta = \pi$

thus: $$t_r = \frac{\pi - \theta}{\omega_d}$$

It indicates that when $\zeta$ is fixed, the system response is more rapidly as underdamping natural frequency $\omega_n$ increases.( $\omega_d = \omega_n\sqrt{1-\zeta^2}$ )

# Summary

$$t_P = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$\sigma_P = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

$$t_s \approx \frac{3\sim 4}{\zeta\omega_n}$$

- Damping ratio decrease: the rise time declines; the setting time, the overshot and the steady-state error all increase (***$K_v$*** decreases).
- Damping ratio increase: the rise time increases;
- Expectation: short rise time and setting time, small overshot. The damping ratio is usually chosen as 0.4-0.8 in engineering and 0.707 is known as the best damping ratio.

## 2. Critical damping $\zeta = 1$

- Since $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ , we note that the system has two equal real roots

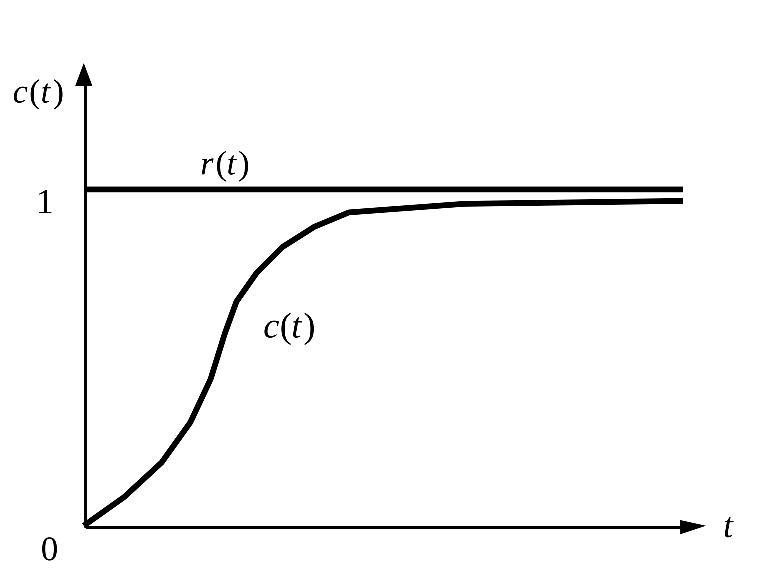
$$s_{1,2} = \pm\omega_n$$

- As for a step input, the Laplace transform of the output is:

$$c(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

$$c(t) = 1 - e^{-\zeta\omega^{n}t}(1 + \omega_n t)$$

- output response:



- **2. Overdamping** $\zeta > 1$
- We note that the system has two unequal real roots:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- As for a step input, the Laplace transform of the output in partial fractions is:

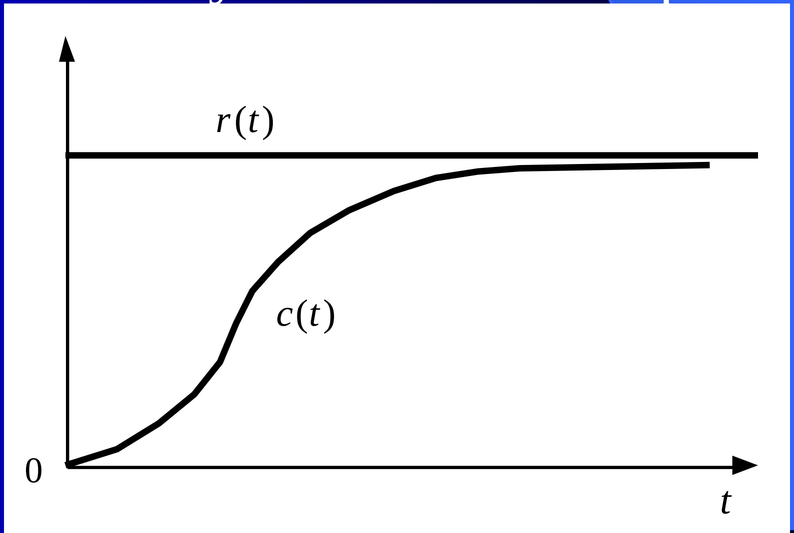
$$c(s) = \frac{1}{s} + \frac{[2(\zeta^2 - \zeta\sqrt{\zeta^2-1} - 1)]^{-1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}} + \frac{[2(\zeta^2 + \zeta\sqrt{\zeta^2-1} - 1]^{-1}}{s + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}}$$

- Apply inverse Laplace transform to the equation above and we have the time domain response when the system is overdamped.

$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2-1}}\left(\frac{e^{T_1 t}}{-T_1} - \frac{e^{T_2 t}}{-T_2}\right)$$

- output response:

$$T_1 = -(\zeta + \sqrt{\zeta^2-1})\omega_n$$
$$T_2 = -(\zeta - \sqrt{\zeta^2-1})\omega_n$$



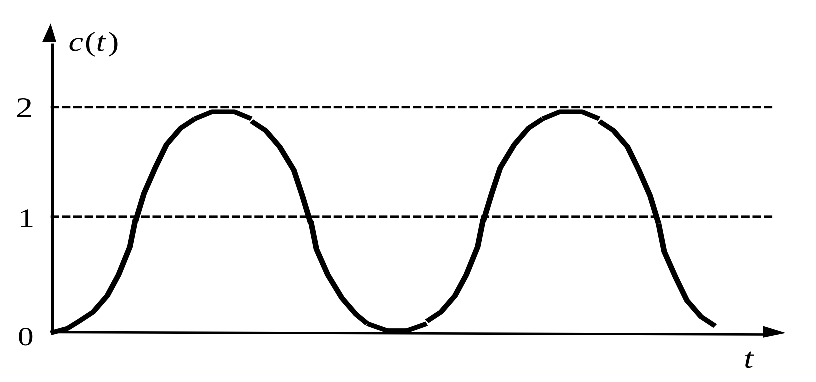
## 4. Undamping $(\zeta = 0)$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

The system has a pair of virtual roots :

$$s_{1,2} = \pm j\omega_n \qquad (\zeta = 0)$$

$$c(t) = 1 - \cos\omega_n t \qquad (t \geq 0)$$




This is a constant amplitude oscillation curve with the average 1.

# 3.6.3 High Order System

- we often take a system higher than third-order as a high-order system. It's usually approximated as a second-order system.
- Consider a control system with a closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{M(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots\cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots\cdots + a_1 s + a_0}$$

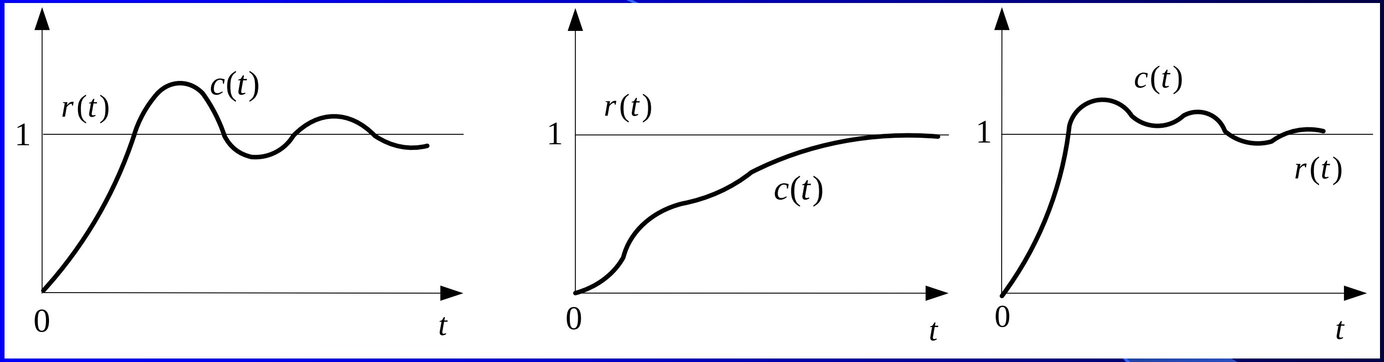
- Step Response is :

$$C(s) = \frac{k\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}(s^2+(\sigma+j\omega_k)(s^2+(\sigma-j\omega_k)} \cdot \frac{1}{s}$$

$$C(t) = a_0 + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{i=1}^{r} e^{-\zeta_k \omega_k t}(B_k \cos \omega_k \sqrt{1-\zeta_k^2}\, t + C_k \sin \omega_k \sqrt{1-\zeta_k^2}\, t)$$

$$= a_0 + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{i=1}^{r} D_k e^{-\zeta_k \omega_k t} \sin(\omega_k \sqrt{1-\zeta_k^2}\quad t + \varphi_k)$$

- **High-order system step response:**




- 1. Response type (in the case of oscillation) is determined by the features of the closed-loop poles.
- 2. The shape of dynamic response curve is determined by both the closed-loop zeros and poles.
- 3. The closed-loop poles affect system characteristics more when they are closer to the imaginary axis.
- Dominant Pole: The closed-loop poles closest to the imaginary axis dominate the dynamic response.
- Dipole: zeros and poles in the same location or quite close with each other affect the dynamic response very little.

**EX: Consider a closed-loop system with the transfer function** $\frac{C(s)}{R(s)} = \frac{312000}{(s+60)(s^2+20s+5200)}$

Closed poles are: $p_{1,2} = -10 \pm j71.7, \quad p_3 = -60$

The ratio of the real part of $P_1$ and $P_2$ to the real part of $P_3$ is :

$$\frac{\mathrm{Re}[p_1]}{\mathrm{Re}[p_3]} = \frac{-10}{-60} = \frac{1}{6} < \frac{1}{5}$$

Thus $P_1$ and $P_2$ is a pair of dominant poles and the step response is :

$$c(t) = 1 - 0.696e^{-10t}\sin(71.7t + 26.93^{o}) - 0.686e^{-60t}$$

Ignore the dynamic component correlated with $P_3$ , the two solutions are close to each other.

$$c(t) = 1 - 0.696e^{-10t}\sin(71.7t + 26.93^{o})$$

# Homeworks

- Page120
  - 3.27
  - Deadline Oct.22.2012