Principles of Automatic Control

-Chap4 Root Locus of Basic Feedback System

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Chapter 4 Root Locus of Basic Feedback System

4-1 Root locus of a basic feedback system
4-2 Guidelines for sketching Root Locus

4-3 System Analysis Using the Root Locus

You can find this content in Chap 5 of our book

What is Root Locus?

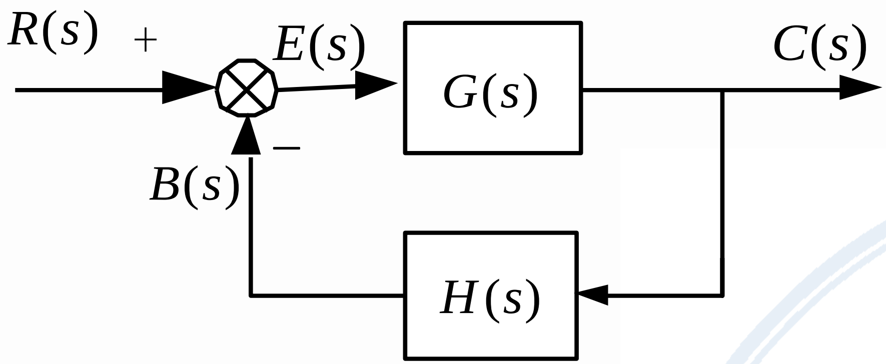
- **W.R Evans (1948, Graphical analysis of control system 控制系统的图解分析)developed a method and rules for plotting the paths of roots, which was called as the Root Locus.**
- **Root Locus is an engineering method of graphically solving the characteristic equation.**
- **With the development of MATLAB and similar software, the rules are no longer need for detailed plotting but it is essential for we to understand how proposed dynamic compensations will influence a locus.**
- **It is basic requirement for us to sketch a locus as a guide in the design process.**

4-1 Root locus of a basic feedback system

- Root locus is the track of the closed-loop poles on s plane varied according to the variation of system parameter(s).
- Also: The root locus is most commonly used to study the effect of loop gain variations.

4-1 Root locus of a basic feedback system

- We begin with the basic feedback system shown as follows:



The closed-loop transfer function is

$$\Phi(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

4-1 Root locus of a basic feedback system

- **The characteristic equation:**

$$1+G(s)H(s)=0$$

whose roots are the poles of this transfer function.

we can get: $G(s)H(s)=-1$

If we define the $G(s)H(s)$ **with poles and zeros, there are :**

$$G(s)H(s)=\frac{K_r\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

4-1 Root locus of a basic feedback system

Let

$$G_k(s)H_k(s)=\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

the $G(s)H(s)$ with poles and zeros can defined as:

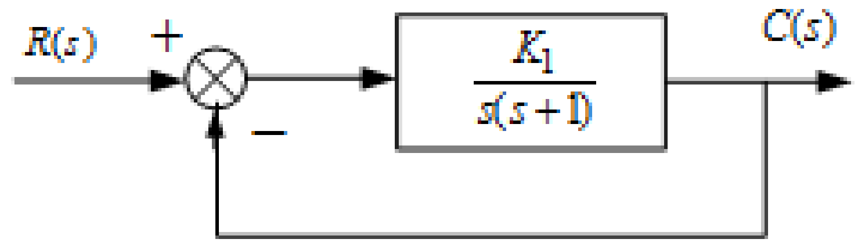
$$G(s)H(s)=\frac{K_r\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=K_rG_k(s)H_k(s)=-1$$

That is : $G_k(s)H_k(s)=-\frac{1}{K_r}$ **OR** $\prod_{j=1}^{n}(s+p_j)+K_r\prod_{i=1}^{m}(s+z_i)=0$

They are the root-locus form or Evans forms of a characteristic equation.

4-1 Root locus of a basic feedback system

Example:



Open-loop transfer function: $$G(s)=\frac{K_1}{s(s+1)}$$

closed-loop transfer function:

$$\Phi(S)=\frac{G(s)}{1+G(s)H(s)}=\frac{G(s)}{1+G(s)}=\frac{\dfrac{K_1}{s(s+1)}}{1+\dfrac{K_1}{s(s+1)}}=\frac{K_1}{s^2+s+K_1}$$

characteristic equation: $$s^2+s+K_1=0$$

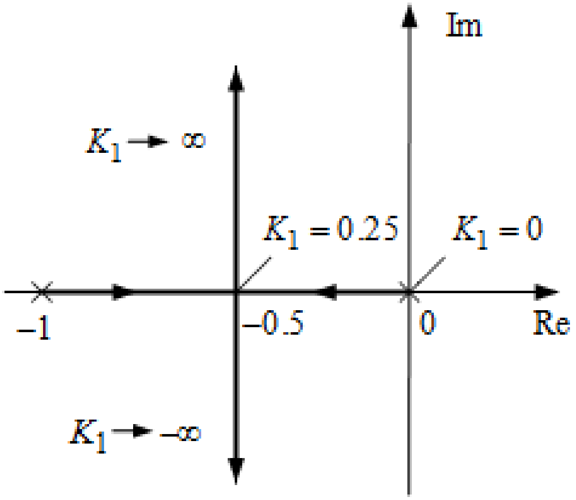
characteristic equation root: $$s_{1,2}=-0.5\pm 0.5\sqrt{1-4K_1}$$

4-1 Root locus of a basic feedback system

As K_1 increases from zero toward infinity, the changes of characteristic equation root are at the list blow.

K_1	0	0.125	0.25	0.5	……	∞
s_1	0	-0.146	-0.5	-0.5+j0.5	……	-0.5+j∞
s_2	-1	-0.854	-0.5	-0.5-j0.5	……	-0.5-j∞

As the value of K_1 increases from zero toward infinity, the corresponding location of s “moves” from an open-loop pole toward infinity.



4-1 Root locus of a basic feedback system

Stability : As the value of K_1 increases from zero toward infinity, Root locus do not cross the imaginary axis to the right hemisphere of s plane, so the system for all values are stable.

Steady-state: Open-loop transfer function in the coordinate origin have a pole, so it belongs to type I system . The root locus value is Kv. If ess is known ,we can sure allowed values range of the close-loop poles.

Dynamic characteristics:

0< K_1 <0.5, close-loop poles are located in the real axis, **damping state(阻尼态)**

K_1 =0.5 , Two closed-loop real poles overlap, **critical damping（临界阻尼）**

K_1 >0.5 , The closed-loop system has complex poles, **owe damping state （欠阻尼）**

4-1 Root locus of a basic feedback system

- Analysis shows that, root locus and system performance have a close relation. However, for the high order system, the analytic method of system root locus figure doesn’t apply. We hope that we can have simple graphic method. According to the known open-loop transfer function we can quickly draw the closed-loop system root locus.
- For this, we need to research the relationship of the open loop zero, pole and the root locus.

4-1 Root locus of a basic feedback system

Advantages of the Evans Root Locus method :

- Provide a straightforward method of graphically "solving" the characteristic equation on the complex plane
- Provide much more insight into the time-domain response of the system
- Provide a powerful design capability, something not obtainable from a direct solution of the characteristic equation

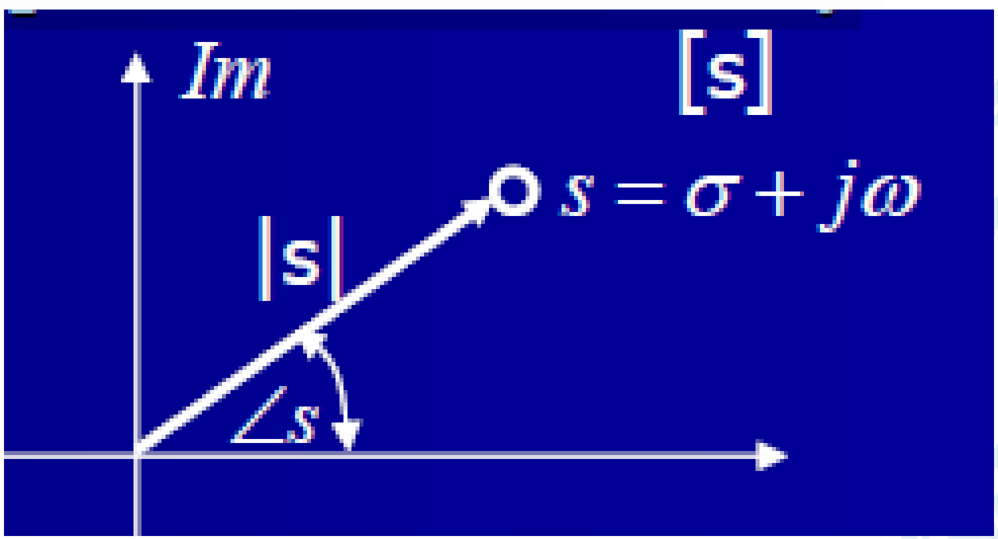
$$1+G(s)H(s)=0$$

4-1 Root locus of a basic feedback system

For a trial value of s, it must satisfy the both equations simultaneously

$\lvert GH(s) \rvert = 1$	Magnitude equation
$\angle GH(s) = -180^\circ$	Argument equation

→ **Phase equation**



4-2 Guidelines for sketching Root Locus

In this section the concepts outlined previously will be developed further into some straightforward guidelines for plotting more complex root locus, which will be illustrated by focusing on a specific example.

<u>Root locus:</u> *(review of previous lecture)*

$$1+G(s)H(s)=0$$

$$G(s)H(s)=-1$$

$$|GH|=1 \quad \text{magnitude equation}$$

$$\angle GH=-180^{\circ} \quad \text{phase equation}$$

Rule #1 The Starting Points and the End Points of the Root Locus (根轨迹的起点和终点)

- The locus **starts** at the open-loop poles (the closed-loop poles for $K = 0$), and **finishes** at the open-loop zeros (the closed-loop zeros for $K=\infty$). The number of segments going to infinity is $n-m$.

(根轨迹始于开环极点，终于开环零点。趋于无穷大的线段条数为n-m。若n>m, 则有n-m条根轨迹终止于无穷远处；若$m>n$，则有 $m-n$条根轨迹起始于无穷远处。)

[Proof]

$$\because \quad G(s)=\frac{K\prod_{j=1}^{m}(s-z_j)}{\prod_{i=1}^{n}(s-p_i)}$$

z_j -- open-loop zero

p_i -- open-loop pole

$$1+G(s)=0 \qquad \prod_{i=1}^{n}(s-p_i)+K\prod_{j=1}^{m}(s-z_j)=0$$

At the starting point of the root locus: K=0

$$\therefore \forall (s-p_i)=0, \quad s=p_i; \quad (i=1,2,\cdots,n)$$

At the end point of the root locus: $K\rightarrow\infty$

and the characteristic equation can be written as

$$\left.\begin{array}{l}\dfrac{1}{K}\prod_{i=1}^{n}(s-p_i)+\prod_{j=1}^{m}(s-z_j)=0\\ \prod_{i=1}^{n}(s-p_i)+K\prod_{j=1}^{m}(s-z_j)=0\end{array}\right\}$$

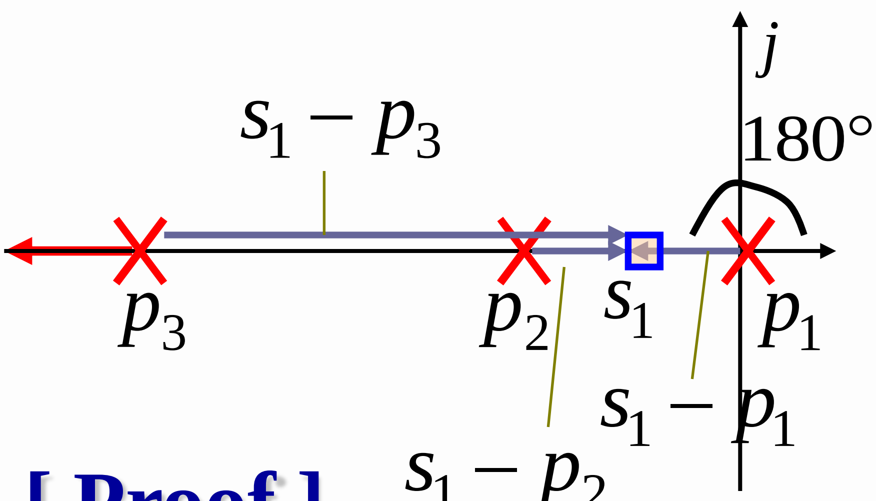
when $K\rightarrow\infty$ $\quad s=z_j$

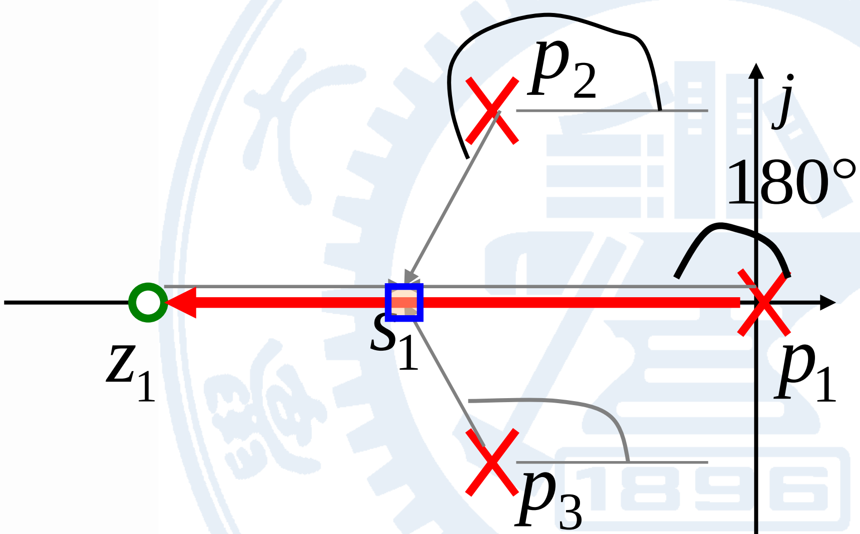
$$(j=1,2,\cdots,m)$$

Rule #2 The Segments of the Root Locus on the Real Axis (实轴上的根轨迹)

- **Segment of the real axis to the left of on an odd number of poles or zeros are segments of the root locus, remembering that complex poles or zeros have no effect.**

(实轴上的根轨迹，是其右侧的开环零、极点数之和为奇数的所在线段。或者说，实轴上，对应零、极点数之和为奇数的左边线段为根轨迹。复数零、极点对该线段没有影响。)





[Proof]

Rule #3 The Symmetry and the Asymptotes of the Root Locus (根轨迹的对称性和渐近线)

- The locus are symmetrical about the real axis since complex roots are always in conjugate pairs.(根轨迹关于实轴对称，因为复数根总是成对出现的。)
- The angle between **adjacent asymptotes** is 360°/(*n*-*m*), and to obey the symmetry rule, the negative real axis is one asymptote when *n*-*m* is odd.(相邻的渐近线之间的夹角是360°/(*n*-*m*), 并同样服从对称规律。当 *n*-*m* 是奇数时，负实轴也是一个渐近线。)
- The Angle of the asymptotes and real axis is：(渐近线与实轴正向的夹角是)

$$\varphi_a = \frac{2k+1}{n-m}\pi \qquad (k = 0, 1, \cdots, n-m-1)$$

Rule #4 The Real Axis intercept of the Asymptotes (渐近线和实轴的交点)

- The asymptotes intersect the real axis at σ_a ,

$$\sigma_a = \frac{\sum p_i - \sum z_j}{n-m}$$

where $\sum p_i$ is the sum of the real parts of the open-loop poles (including complex roots) and $\sum z_j$ is the sum of the real parts of the open-loop zeros (also including complex zeros).

（渐近线与实轴的交点是 $\sigma_a = \dfrac{\sum p_i - \sum z_j}{n-m}$ ，式中 $\sum p_i$ 是开环极点的实部的和（包括复数极点）；$\sum z_j$ 是开环零点的实部的和（包括复数零点）。）

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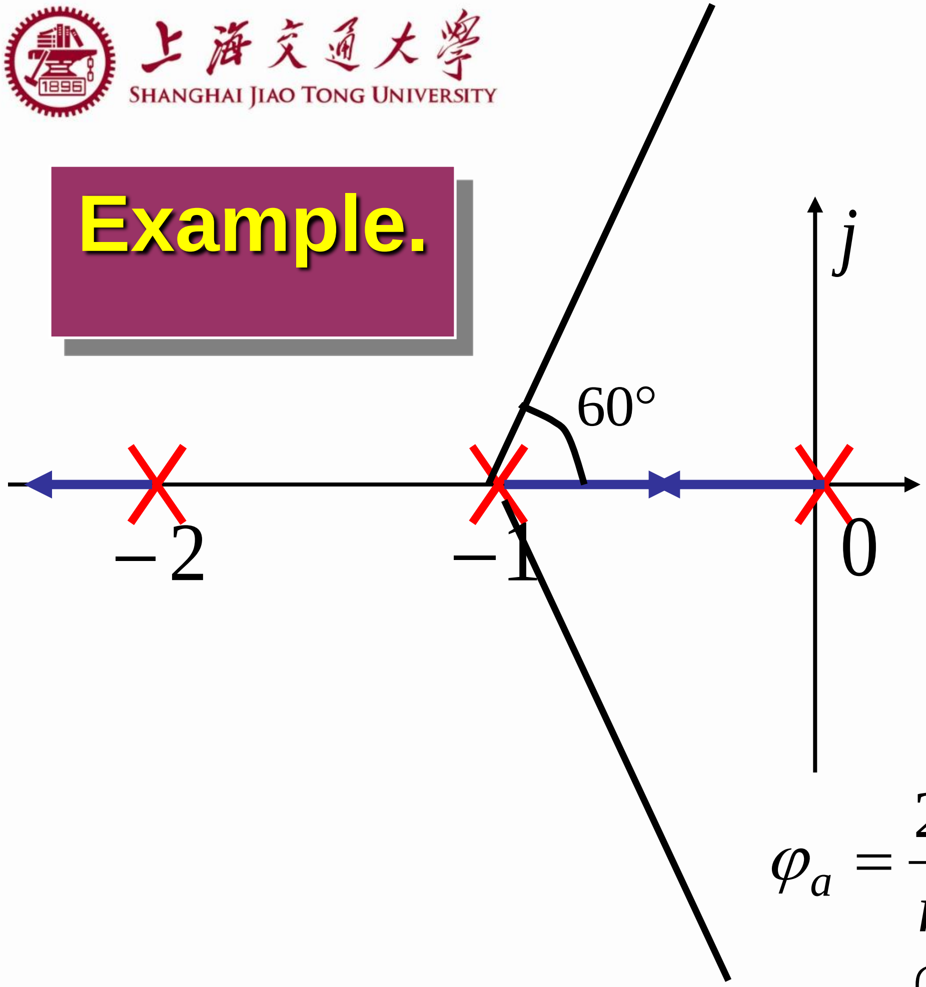
Example.



$$\sigma_a = \frac{\sum p_i - \sum z_j}{n-m} = \frac{0+(-1)+(-2)}{3-0} = -1$$

$$\varphi_a = \frac{2k+1}{n-m}\pi \qquad (k = 0, 1, \cdots, n-m-1)$$

$$= \begin{cases} \pi/3; & k=0 \\ \pi; & k=1 \\ \pi/5 = -\pi/3; & k=2 \end{cases}$$

Rule #5 The Angle of Emergence from Complex Poles and The Angle of Entry into Complex Zeros (根轨迹的出射角和入射角)

- The **angle of emergence** from complex poles is given by 180 º– Σ(angles of the vectors from all other open-loop poles to the poles in equation) + Σ(angles of the vectors from the open-loop zeros to the complex pole in equation). $$\theta_{p_i} = 180° - \sum_{\substack{j=1 \\ j\neq i}}^{n} \angle(p_i - p_j) + \sum_{j=1}^{m} \angle(p_i - z_j)$$

- The **angle of entry** into a complex zero may be found from the same rule and then the sign changed to produce the final result.

$$\varphi_{z_i} = 180° - \sum_{\substack{j=1 \\ j\neq i}}^{m} \angle(z_i - z_j) + \sum_{j=1}^{n} \angle(z_i - p_j)$$

Example.

$$GH(s)=\frac{K(s+5)}{s(s^2+4s+8)}=\frac{K(s+5)}{s(s+2-j2)(s+2+j2)}$$

open-loop poles

$$\theta_{-2+j2}=180^\circ-\boxed{(\angle(-2+j2-0)+\angle((-2+j2)-(-2-j2))}$$

$$+\boxed{\angle((-2+j2)-(-5))}$$

open-loop zeros

$$=180^\circ-tg^{-1}\frac{2}{-2}-tg^{-1}\frac{4}{0}+tg^{-1}\frac{2}{3}$$

$$=180^\circ-135^\circ-90^\circ+33^\circ=-12^\circ$$

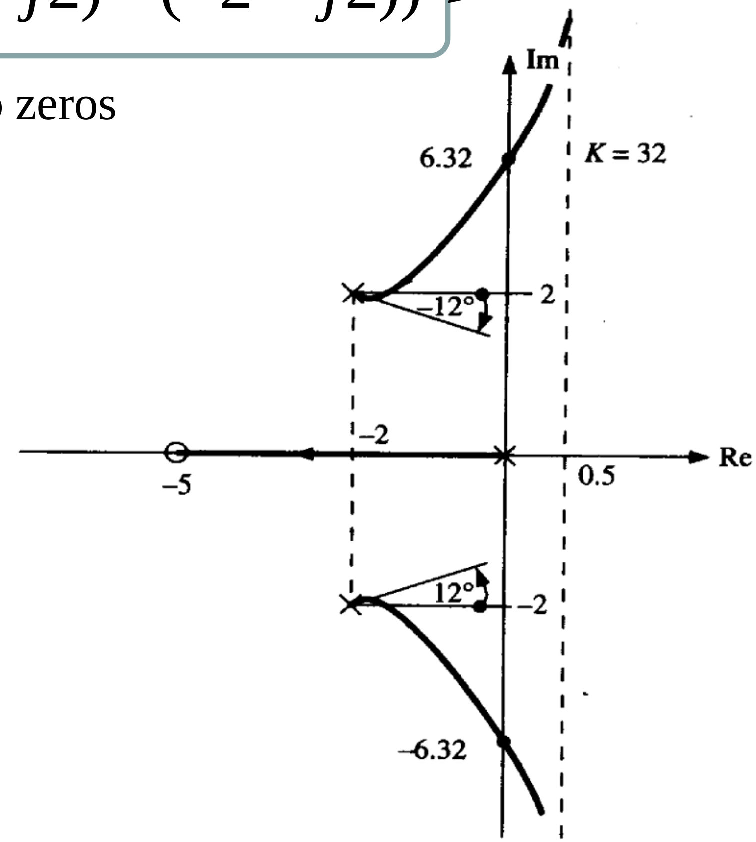


Fig. 10.4 Imaginary-axis crossing point

Rule #6 The Root Locus Crossing with the Imaginary Axis (根轨迹与虚轴的交点)

- The point where the locus crosses the imaginary axis may be obtained by sunbstituting $s = j\omega$ into the characteristic equation and solving for ω.

$$1 + GH(s) = 0 \quad \Rightarrow \quad 1 + GH(j\omega) = 0$$

$$\mathrm{Re}[1 + GH(j\omega) = 0]$$

$$\mathrm{Im}[1 + GH(j\omega) = 0]$$

Example.

$$GH(s) = \frac{K(s+5)}{s(s^2+4s+8)}$$

$$s^3 + 4s^2 + s(K+8) + 5K = 0$$

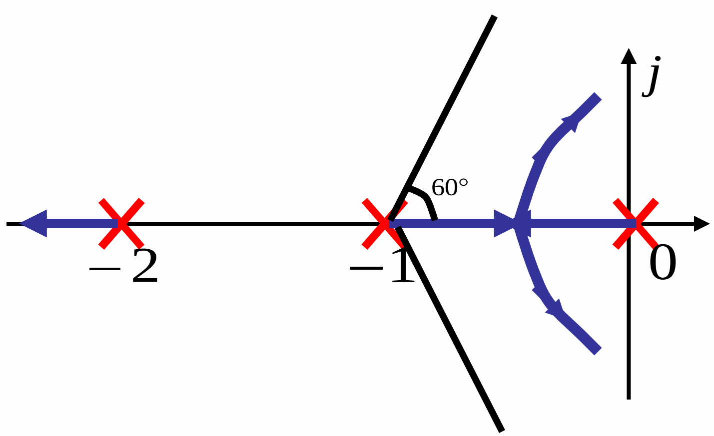
$$(5K - 4\omega^2) + j[(K+8)\omega - \omega^3] = 0$$

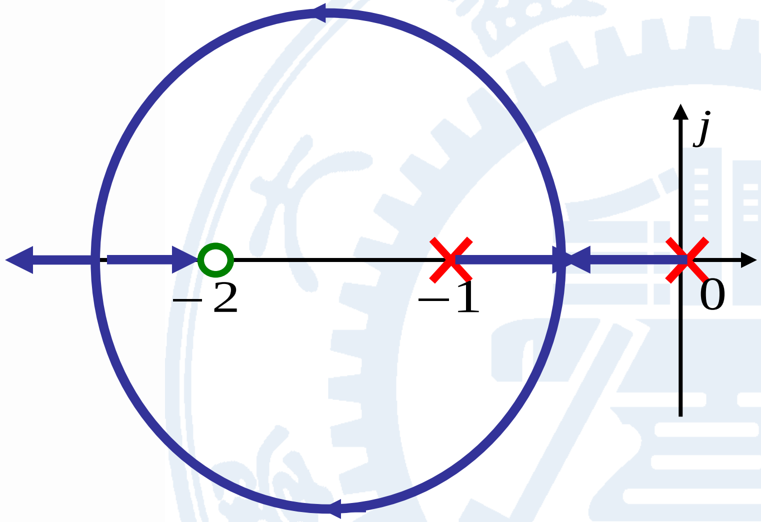
$$\Rightarrow \quad 5K - 4\omega^2 = 0, \quad (K+8) - \omega^2 = 0$$

$$\Rightarrow \quad K = 32, \quad \omega = \pm 6.32$$

Rule #7 The Breakaway Point of the Root Locus (根轨迹的分离点)

- The point at which the locus leaves a real-axis segment is found by determining a local maximum value of *K*, while the point at which the locus enters a real-axis segment is found by determining a local minimum value of *K*.(根轨迹离开实轴区段的点（分离点）由该区段的最大*K*值来确定；而根轨迹进入实轴区段的点（分离点）由该区段的最小*K*值来确定。)





Assume the breakaway point $s = d$:

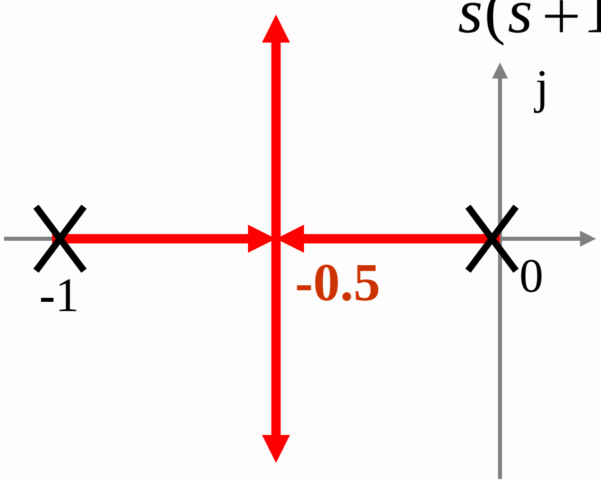
$$(1)\quad \left.\frac{d}{ds}K\right|_{s=d}=0 \quad \left(GH(s)=\frac{K\prod_{j=1}^{m}(s-z_j)}{\prod_{i=1}^{n}(s-p_i)}=-1;\quad \therefore K=-\frac{\prod_{i=1}^{n}(s-p_i)}{\prod_{j=1}^{m}(s-z_j)} \right)$$

$$(2)\quad \sum_{j=1}^{m}\frac{1}{d-z_j}=\sum_{i=1}^{n}\frac{1}{d-p_i}$$

Example: $GH(s)=\dfrac{K}{s(s+1)}$

$$(1)\quad \left.\frac{d}{ds}K\right|_{s=d}=\frac{d}{ds}[-s(s+1)]_{s=d}=-(2d+1)=0$$

$$\therefore d=-0.5$$



$$(2)\quad \sum_{j=1}^{m}\frac{1}{d-z_j}=0=\sum_{i=1}^{n}\frac{1}{d-p_i}=\frac{1}{d}+\frac{1}{d+1}$$

$$\therefore d+1+d=0 \quad \therefore d=-0.5$$

Classroom Exercises

SP. $$GH(s) = \frac{K}{s(s+1)(s+2)}$$

volunteer?

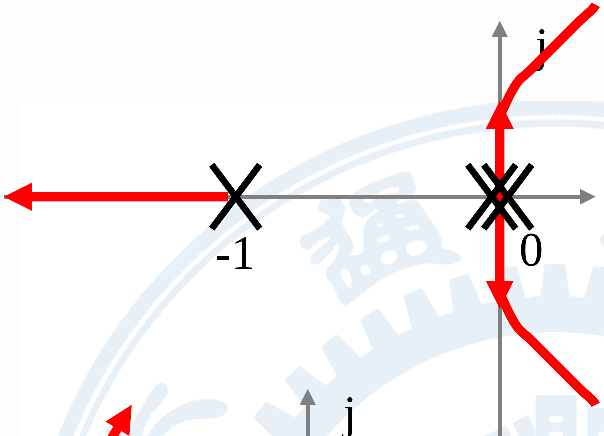
SP. $$GH(s) = \frac{K(s+2)}{s(s+1)}$$

volunteer?

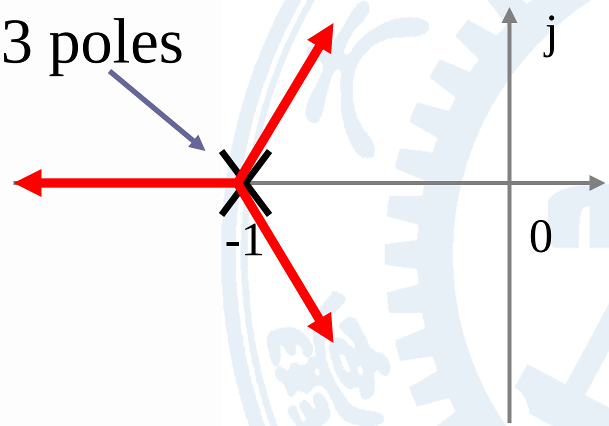
Rule #8 The angle between the direction of emergence (or entry) of q coincident poles (or zeros) on the real axis (根轨迹离开或进入实轴上q重极点（或零点）方向之间的夹角)

$$\psi = \frac{360}{q}$$

SP. $$GH(s) = \frac{K}{s^2(s+1)}$$



SP. $$GH(s) = \frac{K}{(s+1)^3}$$



Rule #9 The gain at a selected point S_t on the locus (在某特定点S_t上的根轨迹增益K)

- The gain at a selected point s_t on the locus is obtained by joining the point to all open-loop poles and zeros and measuring the length of each line $|s_t + p_i|$, $|s_t + z_j|$. The gain is given by

$$K = \left. \frac{\prod_{i=1}^{n} \left| s + p_i \right|}{\prod_{j=1}^{m} \left| s + z_j \right|} \right|_{s=s_t}$$

Sample Problem

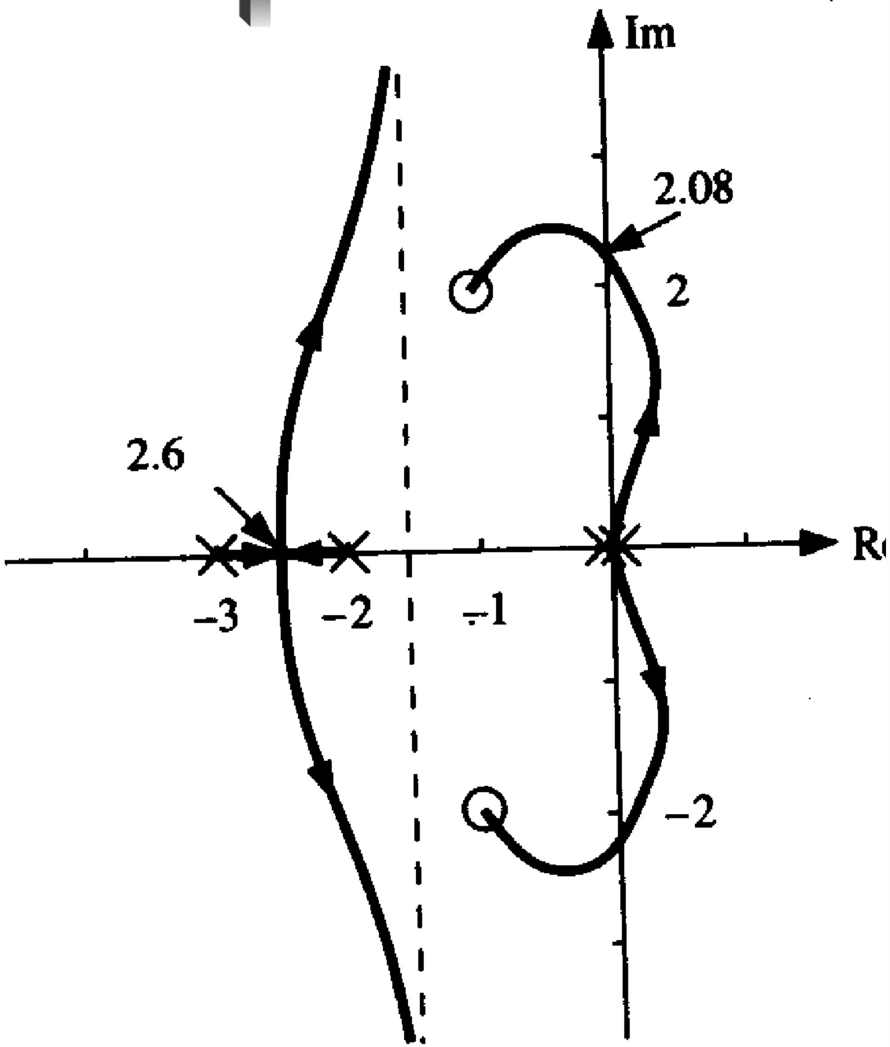


Fig. SP10.1.5

At the breakaway point s = -2.6, Gain K is

$$K = \left.\frac{\prod_{i=1}^{n}|s+p_i|}{\prod_{j=1}^{m}|s+z_j|}\right|_{s=-2.6}$$

$$= \frac{|-2.6+2||-2.6+3|2.6^2}{|-2.6+1+j2||-2.6+1-j2|}$$

$$\approx 0.2473$$

Rule #10 The sum of the closed-loop poles (闭环极点之和)

- **If there are at least two more open-loop poles than open-loop zeros, the sum of the closed-loop poles is constant, independent of *K*, and equal to the sum of the real parts of the open-loop poles.**

(如果开环极点比开环零点至少多2个，闭环极点的和为一不依赖于K的常数，且等于开环极点的实部的和。)

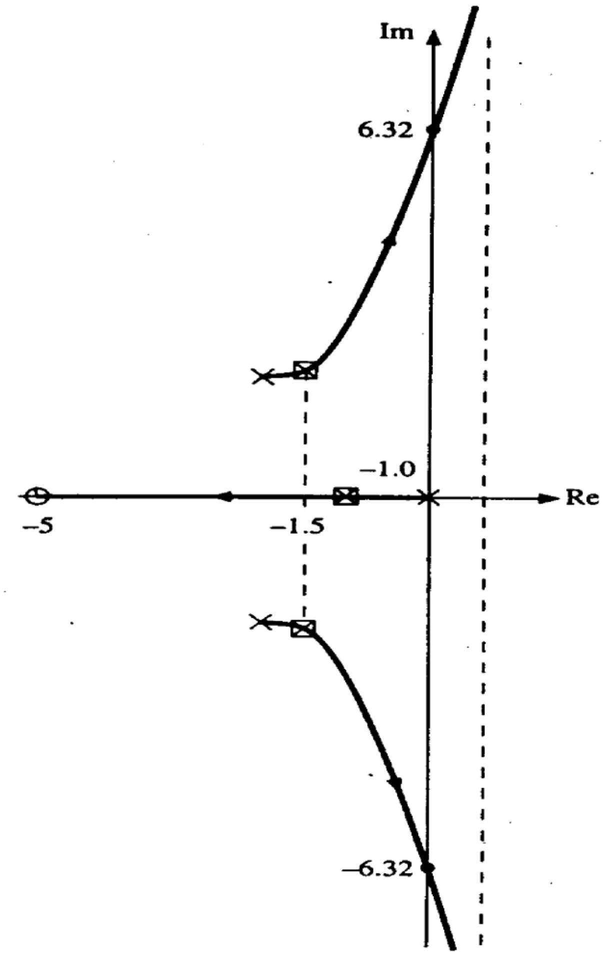


Fig. 10.8 Complete root locus with roots for $K = 5$

$$GH(s) = \frac{K(s+5)}{s(s+2+j2)(s+2-j2)}$$

$$if \quad s_{1,2} = -\sigma \pm j\omega, \quad s_3 = -1 \qquad -1-\sigma-\sigma = 0-2-2$$

$$\therefore \sigma = 1.5$$

Rule #11 The number of branches of the root locus (根轨迹的分支数)

- **The number of branches of the root loci is equal to the maximum in the number *N* of poles and the number *M* of zeros of the open-loop transfer function.**

(根轨迹的分支数等于传递函数中极点数*M*和零点数*N* 中的最大数)

$$b = \max\{N, M\}$$

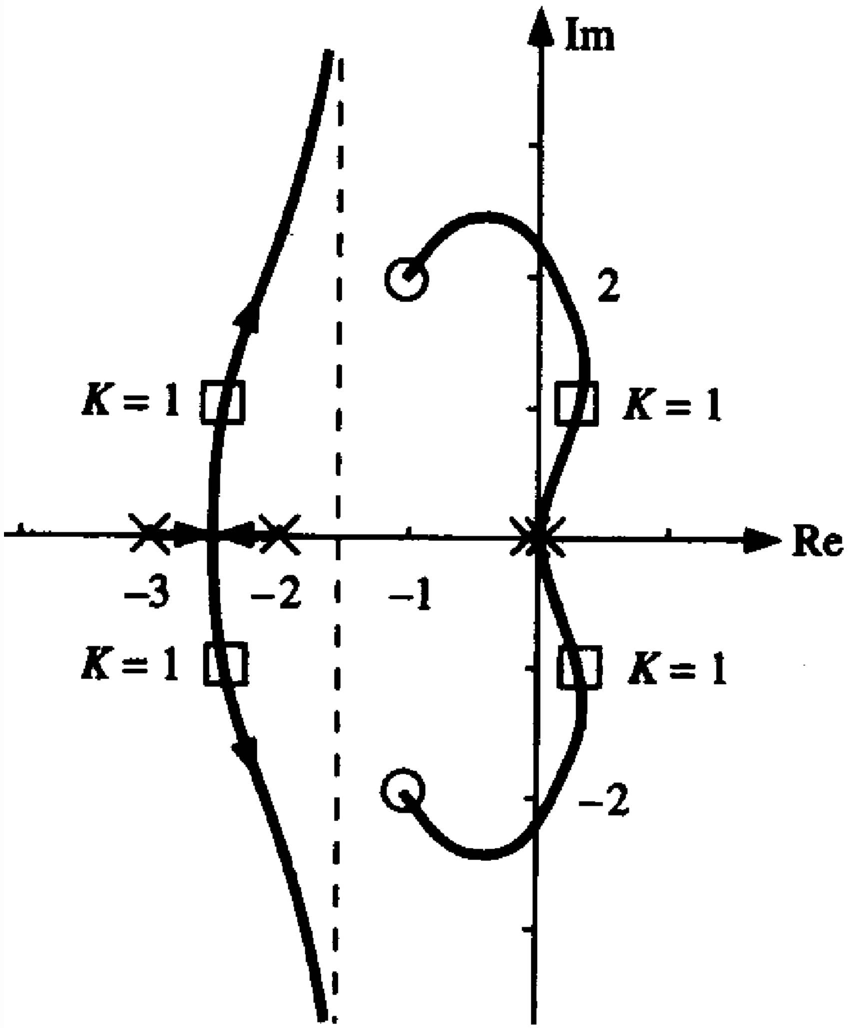


Fig. SP10.1.6

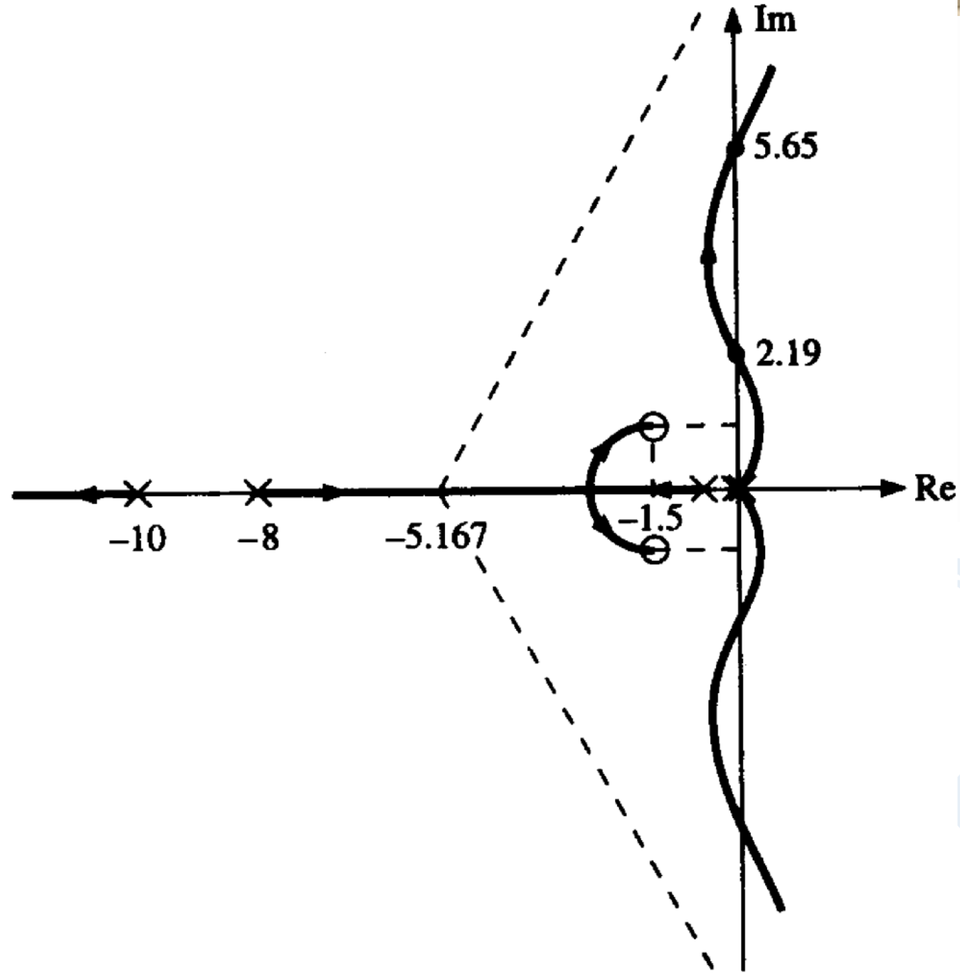


Fig. SP10.2.4

Summary of the Root locus plotting rules (steps * are optional):

Step 1: Mark the poles and zeros with "x" and "o" on the complex plane

Step 2: Draw the locus on the real axis to the left of odd number of real poles and zeros

Step 3: Draw *n-m* asymptotes centered at σ_a and leaving at angles ϕ_l

$$\sigma_a = \frac{\sum p_i - \sum z_i}{n-m} \quad \phi_l = \frac{180+360l}{n-m}; l = 0,1,2,\ldots n-m-1$$

Step 4: Compute departure angles from poles "x" and arrival angles at zeros "o".

Step 5*: Using $s = j\omega$ find the locations where the locus crosses the imaginary axis

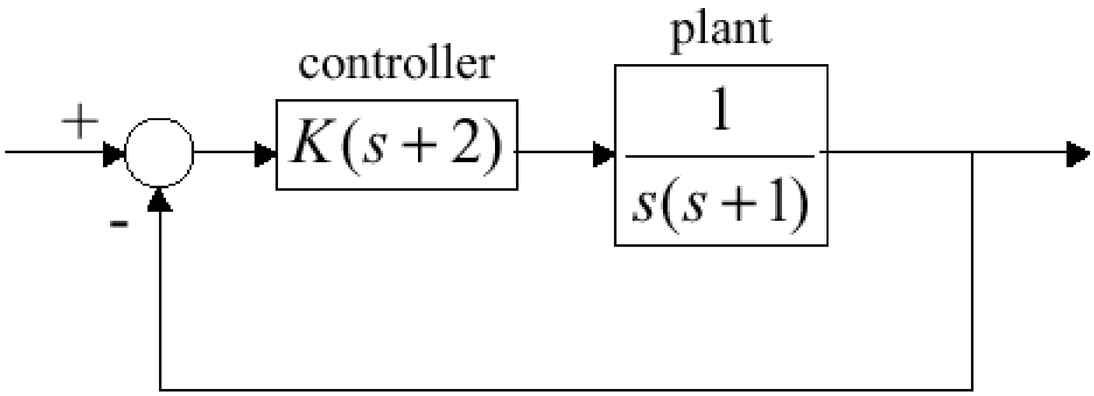
Step 6*: Find any multiple roots on the locus by solving

$$\frac{dK(s)}{ds} = 0 \Rightarrow \text{the solutions } s_0 \text{ are the locations of multiple poles}$$

for s. If these roots are on the real axis, these are either break away or break-in points

Step 7: Complete the root locus using the facts developed in steps 1-6.

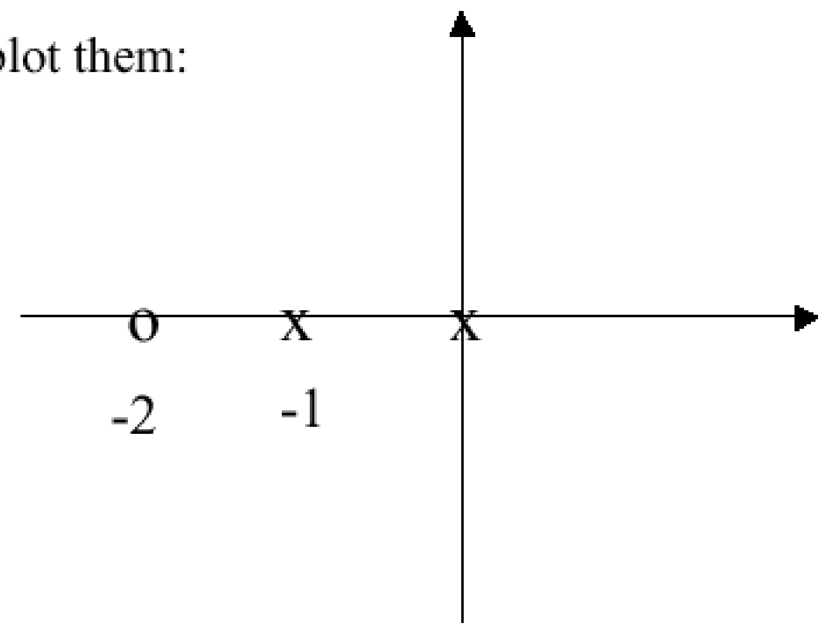
Example 1: Plot the root locus of the following system



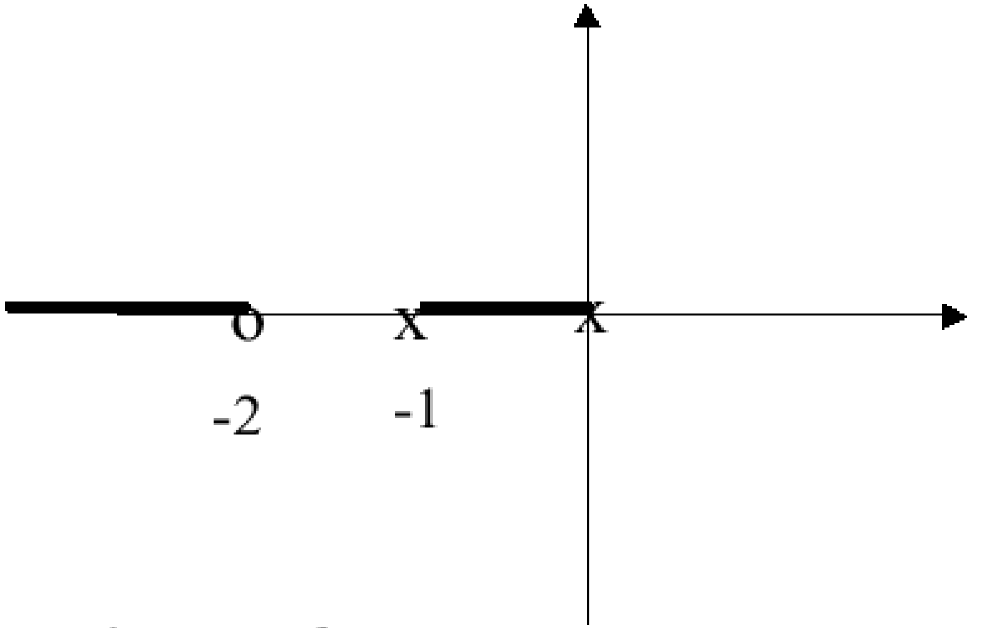
Step one: find the poles and zeros of GH and plot them:

$$GH = \frac{K(s+2)}{s(s+1)}$$

$$p_1 = 0; p_2 = -1; z_1 = -2$$



Step two: draw locus on the real axis



Step three: calculate asymptote angles and center of asymptotes:

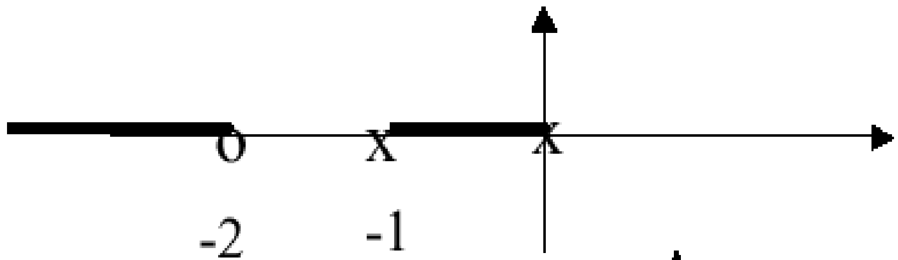
$$\sigma_a = \frac{\sum p_i - \sum z_i}{n-m} = \frac{0+(-1)-(-2)}{2-1} = 1$$

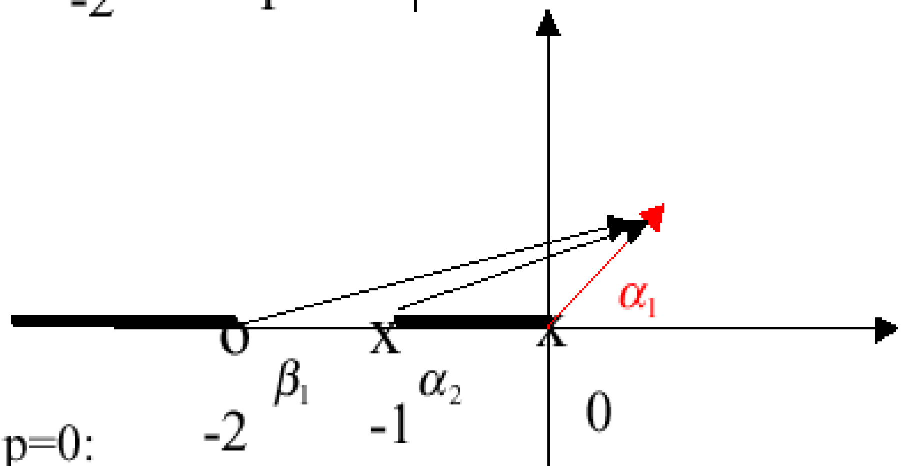
$$\phi_l = \frac{180+360l}{n-m} = \frac{180+360l}{2-1} = 180; l=0,$$

since $n-m=1$ there is only one asymptote at 180°,

corresponding to the real axis

Step four: compute departure angles at poles and arrival angles at zeros

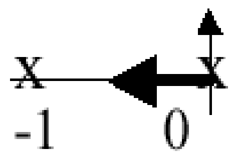


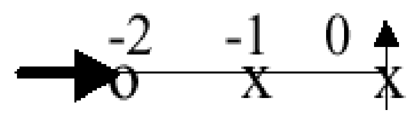


Departure angle at p=0:

$$\beta_1 - \alpha_1 - \alpha_2 = 180$$

$$\beta_1 = 0; \alpha_2 = 0 \Rightarrow \alpha_1 = -180$$

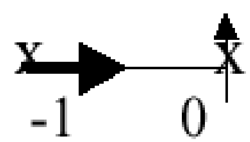




Departure angle at p=-1:

$$\beta_1 - \alpha_1 - \alpha_2 = 180$$

$$\beta_1 = 0; \alpha_1 = 180 \Rightarrow \alpha_2 = -360 = 0$$



Arrival angle at z= -2:

$$\beta_1 - \alpha_1 - \alpha_2 = 180$$

$$\alpha_2 = 180; \alpha_1 = 180 \Rightarrow \beta_1 = 180 + 180 + 180 = 180$$

Step five: find locations on the imaginary axis

$$GH = \frac{K(s+2)}{s(s+1)} = -1$$

substitute $s = j\omega$

$$\frac{K(j\omega+2)}{j\omega(j\omega+1)} = -1$$

$$K(j\omega+2) = -j\omega(j\omega+1)$$

$$jK\omega = -j\omega \Rightarrow K = -1; \omega = 0$$

$$2K = \omega^2 \Rightarrow \text{the only real solution } K = 0$$

Step six: find multiple roots, especially on the real axis:

$$GH = \frac{K(s+2)}{s(s+1)} = -1$$

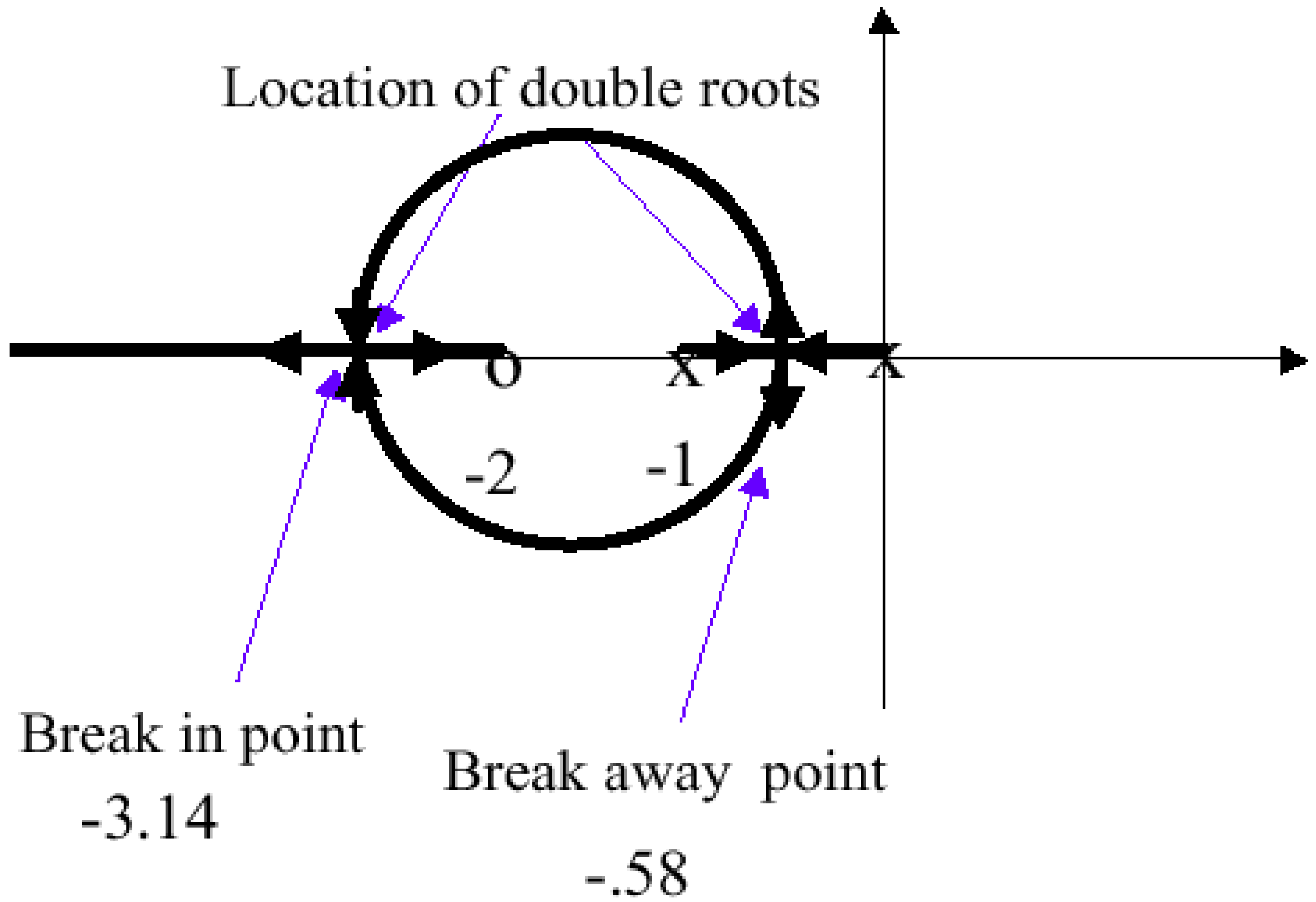
$$K = -\frac{s(s+1)}{(s+2)}$$

$$\frac{dK}{ds} = -\frac{(2s+1)(s+2) - s(s+1)}{(s+2)^2} = 0$$

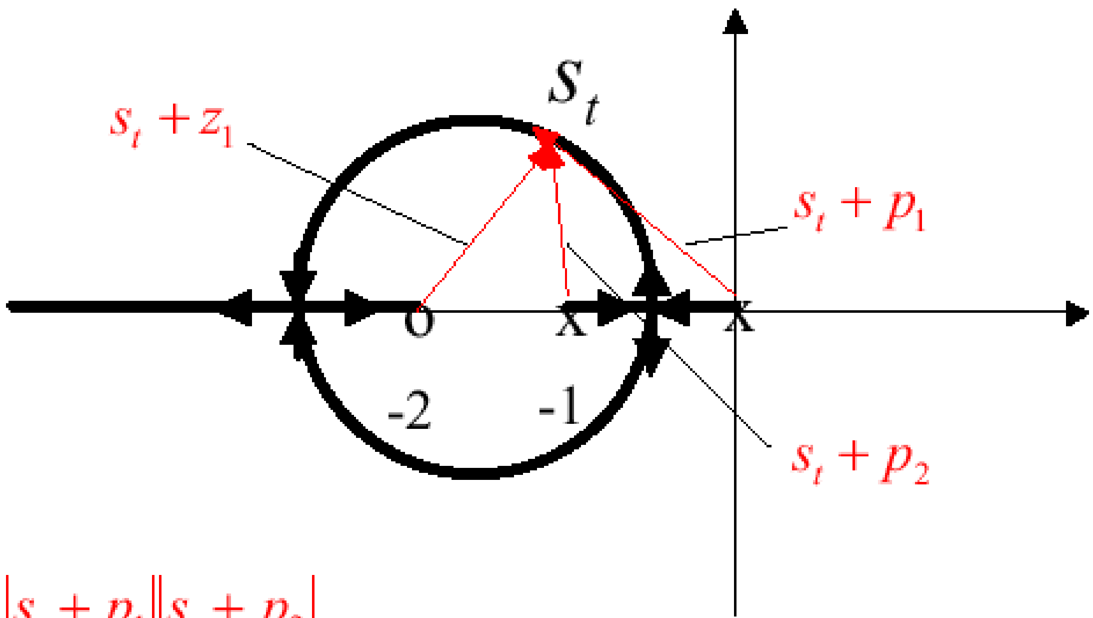
$$2s^2 + 5s + 2 - s^2 - s = 0$$

$$s^2 + 4s + 2 = 0$$

$$s_{1/2} = -2 \pm \sqrt{2} = -0.5857; -3.4142$$

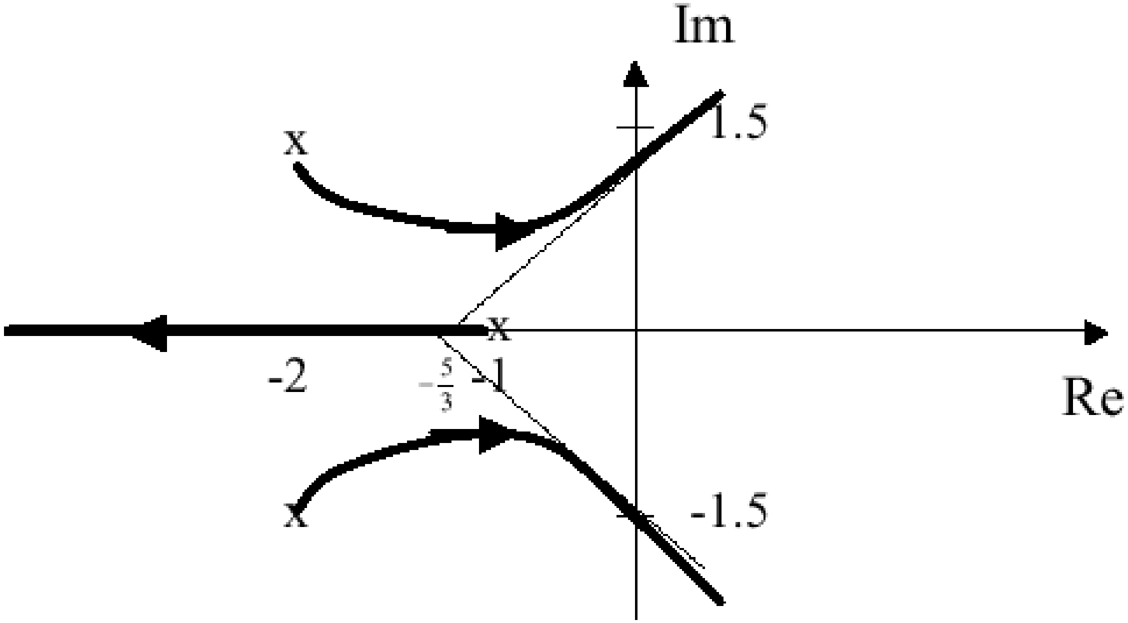

Location of double roots
o
x
x
-2
-1
Break in point
-3.14
Break away point
-.58

Q: How can we determine the value of the gain *K* for a particular pole location?



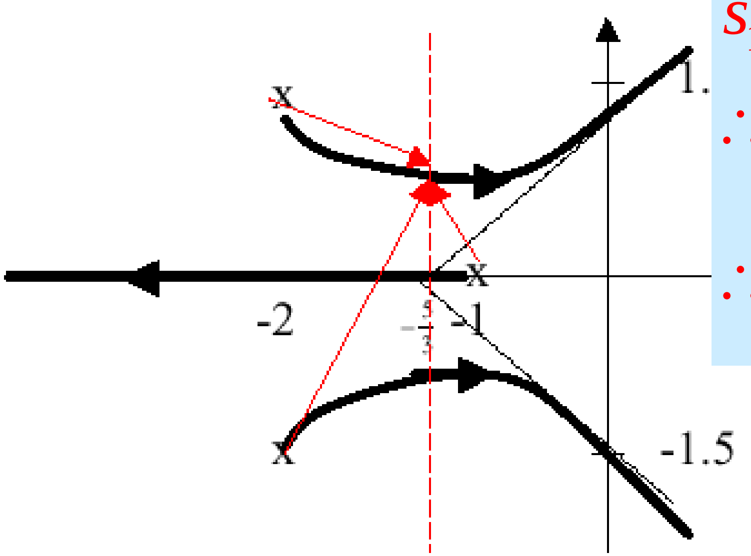
$$K = \frac{|s_t + p_1||s_t + p_2|}{|s_t + z_1|}$$

Example 2: p. 10.5: Given the open loop pole-zero map,find the value of *K*, for which the closed loop poles have equal real parts.



$$\sigma_a = \frac{\sum p_i - \sum z_i}{n-m} = \frac{(-2)+(-2)+(-1)-(0)}{3-0} = -\frac{5}{3}$$

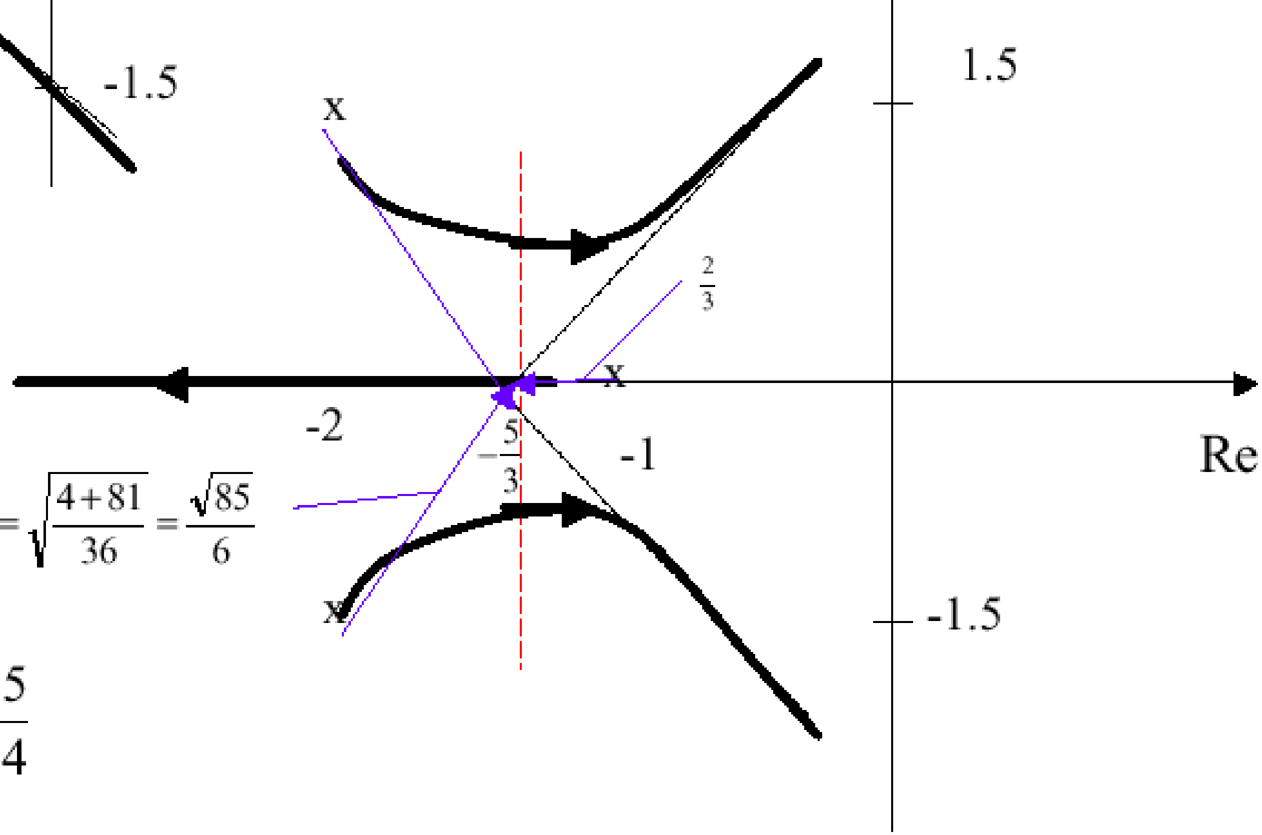
$$\phi_l = \frac{180+360l}{n-m} = \frac{180+360l}{3-0} = 60;180;-60;\ l = 0,1,2$$



$$s_1 = \sigma + j\omega,\ \ s_2 = \sigma - j\omega,\ \ s_3 = \sigma$$

$$\therefore \sum s_i = 3\sigma = \sum p_i = -2 + j\omega - 2 - j\omega - 1$$

$$\therefore \quad \sigma = -\frac{5}{3}$$



$$\sqrt{\frac{1}{3^2} + \frac{3^2}{2^2}} = \sqrt{\frac{4+81}{36}} = \frac{\sqrt{85}}{6}$$

$$K = \frac{\sqrt{85}}{6}\,\frac{\sqrt{85}}{6}\,\frac{2}{3} = \frac{85}{54}$$

System design in the complex plane

$$(1) \quad T_r = \frac{1}{\omega_d}\left(\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \approx \frac{1.8}{\omega_n} \Rightarrow \omega_n \geq \frac{1.8}{t_r}$$

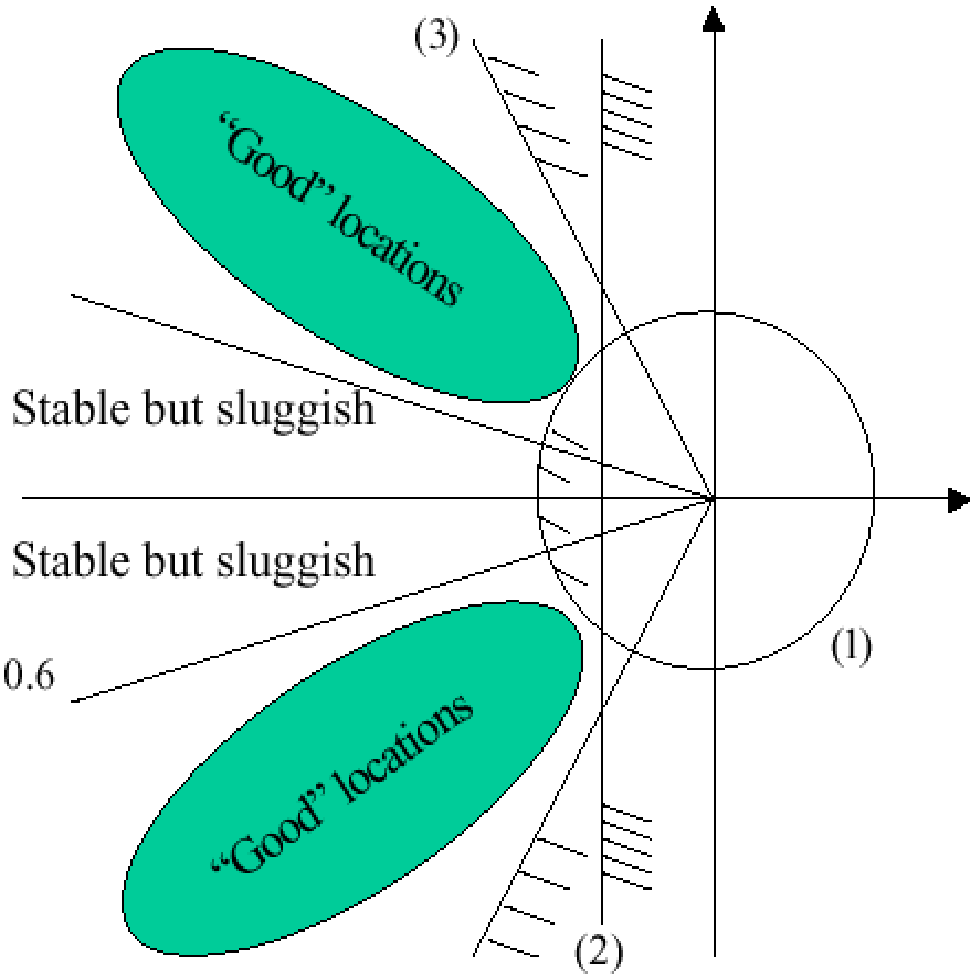
$$(2) \quad T_s \approx \frac{4.6}{\zeta\omega_n} \Rightarrow \zeta\omega_n \geq \frac{4.6}{t_s}$$

$$(3) \quad PO\% = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} \cong 100\left(1 - \frac{\zeta}{0.6}\right) \quad 0 \leq \zeta \leq 0.6$$

$$\Rightarrow \zeta \geq 0.6(1 - PO\%/100)$$

$$(4) \quad e_{ss} = \frac{1}{1+K_p} \text{ or } \frac{1}{K_v} \text{ etc.}$$

$$\Rightarrow K_p \geq \frac{1-e_{ss}}{e_{ss}} \text{ or } K_v \geq \frac{1}{e_{ss}} \text{ etc.}$$



Instructional objectives:
At the end of this lecture students should be able to

- Find the directions of the pole asymptotes
- Find the values of K for which the pole cross the imaginary axis
- Determine plot the root locus and select "good" values for K

Homework

- **Page 223**
  - — Ex1
- **Page224-225-226**
  - — Ex5.2
  - — Ex5.3
  - — Ex5.4(a)/(b)
  - — Ex5.5(a)/(b)
  - — Ex5.6(a)/(b)
  - — Ex5.7(a)/(b)
  - — Ex5.8(a)/(b)

Deadline:5.Nov.2012

- Chap4 Root Locus of Basic Feedback System is an important engineering method.

Principles of Automatic Control
-Chap4 Root Locus of Basic Feedback System(2)
System Analysis Using the Root Locus

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Review for 4-2 Guidelines for sketching Root Locus

	Content	Guidelines
1	Starting point 起点 End Point 终点	The locus starts at the open-loop poles (the closed-loop poles for $K = 0$), and finishes at the open-loop zeros (the closed-loop zeros for $K=\infty$). 起始于开环的极点，终止于开环传的零点（包括无限零点）
2	The number of segments 分支数	The number of segments going to infinity is n-m. 等于开环传递函数的极点数（n≥m）
3	symmetric 对称性	The locus are symmetrical about the real axis since complex roots are always in conjugate pairs. 对称于实轴

Review for 4-2 Guidelines for sketching Root Locus

	Content	Guidelines
4	asymptotes 渐近线	The angle between adjacent asymptotes is 360°/(*n*-*m*), and to obey the symmetry rule, the negative real axis is one asymptote when *n*-*m* is odd. The Angle of the asymptotes and real axis is: (渐近线与实轴正向的夹角是) $$\varphi = \frac{\pm 180^{\circ}(2k+1)}{n-m}$$ The asymptotes intersect the real axis is:(渐近线与实轴的交点) $$\sigma_a = -\frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m}$$

Review for 4-2 Guidelines for sketching Root Locus

	Content	Guidelines
5	The Segments of the Root Locus on the Real Axis 实轴上分布	Segment of the real axis to the left of on an odd number of poles or zeros are segments of the root locus, remembering that complex poles or zeros have no effect. 实轴上的根轨迹在实轴的某一区间内存在根轨迹，则其右边开环传递函数的零点、极点数之和必为奇数

Review for 4-2 Guidelines for sketching Root Locus

	Content	Guidelines
6	The Breakaway Point of the Root Locus 分离（回合）点	The point at which the locus leaves a real-axis segment is found by determining a local maximum value of K, while the point at which the locus enters a real-axis segment is found by determining a local minimum value of K. (实轴上的分离（会合）点 $\frac{d[G(s)H(s)]}{ds}=0$或$\frac{dK}{ds}=0$ ——（必要条件）

Review for 4-2 Guidelines for sketching Root Locus

	Content	Guidelines
7	The angle of emergence 出射角 The **angle of entry** 入射角	The angle of emergence from complex poles is given by 180 º– Σ(angles of the vectors from all other open-loop poles to the poles in equation) + Σ(angles of the vectors from the open-loop zeros to the complex pole in equation). 复极点处的出射角： $$\theta_{p_i}=180^\circ-\sum_{\substack{j=1\\j\neq i}}^{n}\angle(p_i-p_j)+\sum_{j=1}^{m}\angle(p_i-z_j)$$ The **angle of entry** into a complex zero may be found from the same rule and then the sign changed to produce the final result. 复零点处的入射角： $$\varphi_{z_i}=180^\circ-\sum_{\substack{j=1\\j\neq i}}^{m}\angle(z_i-z_j)+\sum_{j=1}^{n}\angle(z_i-p_j)$$

Review for 4-2 Guidelines for sketching Root Locus

	Content	Guidelines
8	The Root Locus Crossing with the Imaginary Axis 虚轴交点	（1）满足特征方程$1+G(j\omega)H(j\omega)=0$的 $j\omega$ 值； （2）由劳斯阵列求得（及K_1响应的值）； The point where the locus crosses the imaginary axis may be obtained by sunbstituting $s = j\omega$ into the characteristic equation and solving for ω.
9	The gain at a selected point s_t on the locus 在某特定点s_t上的根轨迹增益K	The gain at a selected point s_t on the locus is obtained by joining the point to all open-loop poles and zeros and measuring the length of each line $\|s_t + p_i\|$, $\|s_t + z_j\|$. The gain is given by $K = \left.\dfrac{\prod_{i=1}^{n}\|s+p_i\|}{\prod_{j=1}^{m}\|s+z_j\|}\right\|_{s=s_t}$

Review for 4-2 Guidelines for sketching Root Locus

	Content	Guidelines
10	The sum of the closed-loop poles 闭环极点之和	If there are at least two more open-loop poles than open-loop zeros, the sum of the closed-loop poles is constant, independent of *K*, and equal to the sum of the real parts of the open-loop poles. (如果开环极点比开环零点至少多2个，闭环极点的和为一不依赖于*K*的常数，且等于开环极点的实部的和。)
11	The number of branches of the root locus 根轨迹的分支数	The number of branches of the root loci is equal to the maximum in the number *N* of poles and the number *M* of zeros of the open-loop transfer function. （根轨迹的分支数等于传递函数中极点数*M*和零点数*N* 中的最大数）

4-3 System Analysis Using the Root Locus

例: 系统的开环传递函数 $G(s)H(s)=\dfrac{K_1}{s(s+4)(s+6)}$

试画根轨迹，并确定 $\zeta=0.5$ 时K_1的值。

解： 只对根轨迹曲线的特征点进行分析。

(1) 渐近线：3条。

渐近线的夹角： $\varphi=\dfrac{\pm180^\circ(2k+1)}{3-1}=\pm60^\circ,180^\circ$

渐近线与实轴的交点：

$$-\lambda=-\frac{(0+4+6)-0}{3}=-3.33$$

（2）分离点： $\dfrac{1}{s}+\dfrac{1}{s+4}+\dfrac{1}{s+6}=0$

即 $$3s^2+20s+24=0$$

$$s_1=-1.57 \qquad s_2=-5.1 \text{（舍去）}$$

4-3 控制系统性能的复域分析

（3）与虚轴的交点

系统的特征方程：s(s+4)(s+6)+K_1=0

令 $s=j\omega$代入，求得

实部方程: $10\omega-K_1=0$

虚部方程：$\omega^3-24\omega=0$

解得：
$$\begin{cases}\omega=\pm 4.9\\ K_1=240\end{cases}\qquad \begin{cases}\omega=0\\ K_1=0\end{cases}\text{(舍去)}$$

（4）确定 $\zeta=0.5$时的K_1 值： 过原点作OA射线交根轨迹于A，

使得 $\angle AOC=\cos^{-1}0.5=,60^{\circ}$测量得:

$$OA=2.4,\ AB=5.3,\ AC=3.5$$

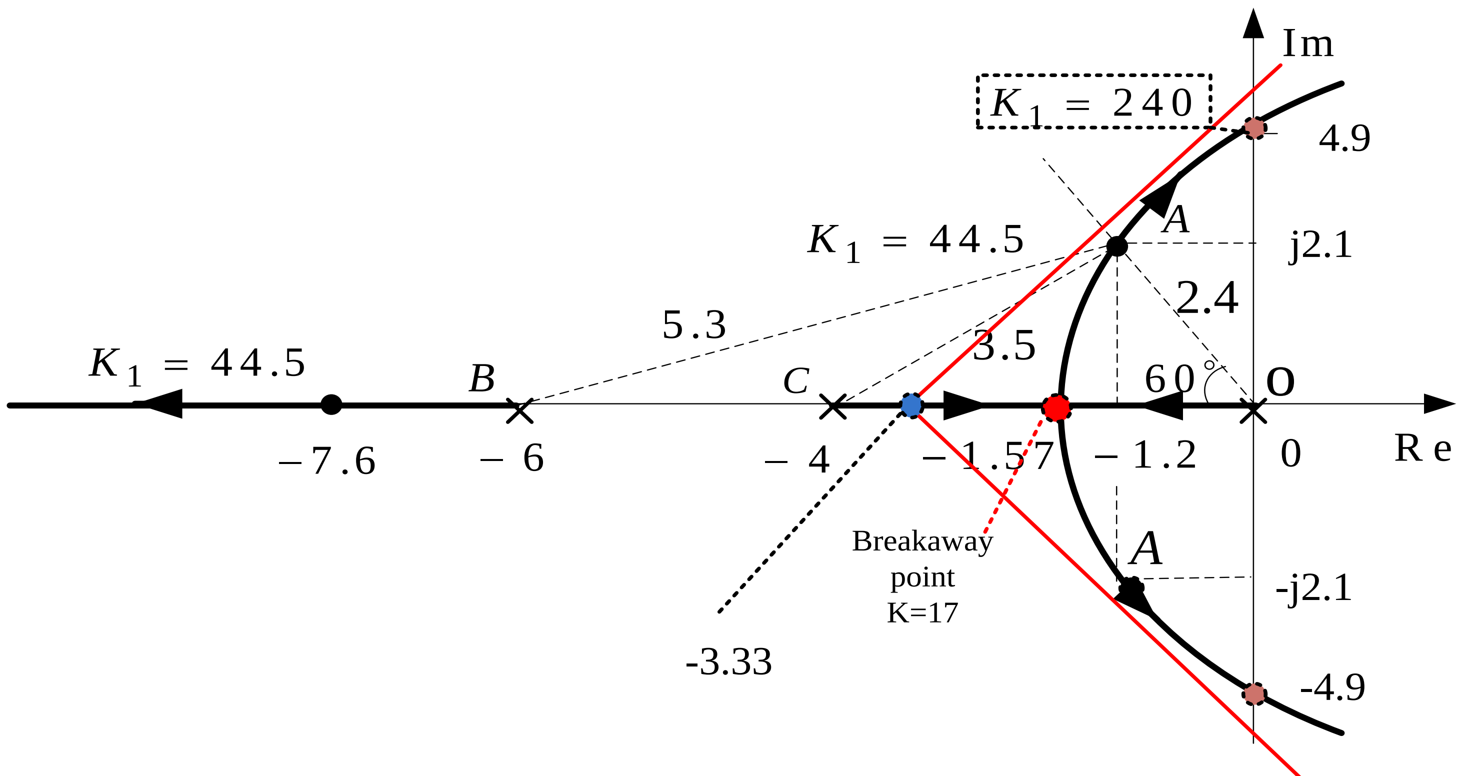
求得
$$K_1=\frac{2.4\times3.5\times5.3}{1}=44.5$$

4-3 System Analysis Using the Root Locus

A点对应的坐标，即闭环的一个极点位置: $s_1 = -1.2 + j2.1$

K_1=44.5时另外两个极点 $s_2 = -1.2 - j2.1$ $s_3 = -7.6$

同理可求得根轨迹在实轴上的分离点-1.57处对应的K_1=17。



4-3 System Analysis Using the Root Locus

A complex domain analysis(复域分析)

- 1. The stability analysis（稳定性分析）:
  - When K1 = 240, has a pair of virtual root, a critical stability, output amplitude oscillation.(临界稳定，输出等幅振荡)
  - When K1 > 240, root locus curve to S right plane, there are a pair of system is the real part of conjugate root, so the system in a state of instable.(系统不稳定)
  - When K1 < 240, the system of the real root section are negative, namely the root is distributed in S left brain plane, system is stable .(系统稳定)

4-3 System Analysis Using the Root Locus

A complex domain analysis(复域分析)

- 2. Steady state performance analysis（稳态性能分析）
  - The system open loop roots tracing gain K1 ring opening amplification coefficient is proportional to the stability of the system, and so on, for the greater the K1, the smaller the steady-state error, steady state performance and the better, but the end is K1 not greater than 240, otherwise, the system will appear not steady state.

4-3 System Analysis Using the Root Locus

3. Dynamic performance analysis（动态性能分析）

- When 0 < K1 more than 17, the system is the root of the negative real number, at this point, as three inertia link in series, the system output have aperiodic characteristics（系统输出具有非周期特性）.
- When 17 < K1< 240, the system has two root locus branch into complex plane, produce a pair of conjugate root, make the order of system step response with the characteristics of oscillation. Along with the increase, the closer the triassic-paleogene root virtual axis, the more powerful output oscillation.

If take ζ = 0.5, poles: s1 = 1.2 + j2.1, s2 = 1.2-j2.1, s3 = 7.6. Due to the relatively s1, s2, s3 is away from the imaginary axis, s1, s2 can be thought of as dominant pole ,and s3 is negligible, can with the second order system's dynamic index to approximate the actually three order system. It is not difficult to get the system dynamic performance index of the complex domain: ζ = 0.5, ω n = 2.4 corresponding to the time domain indexes (忽略S3，三阶系统可以近似为二阶系统), dynamic performance：

$$t_s = \frac{3.5}{\zeta\omega_n} = 2.9s \qquad \sigma_p = 16.3\%$$

4-3 System Analysis Using the Root Locus

the effect the performance of the system when add the open loop zero and pole

The integral pattern of system root locus are joint defined by open loop transfer function of zero and the poles. The open loop zero, pole position, root locus have quite difference.

◆**1. Add poles (with specific system to illustrate)**

General speaking, when the function G (s) H (s) in s left plane, to add poles, the original root locus move to right way, stability decline.

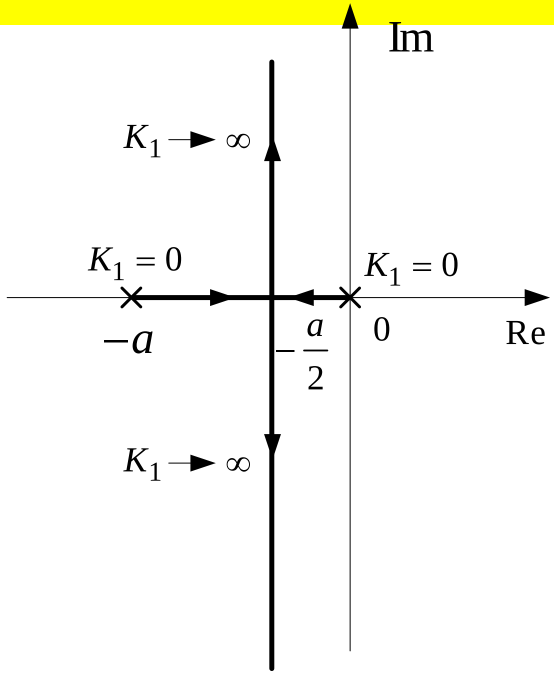
A system of open-loop transfer function:

$$G(s)H(s)=\frac{K_1}{s(s+a)} \quad (a>0)$$

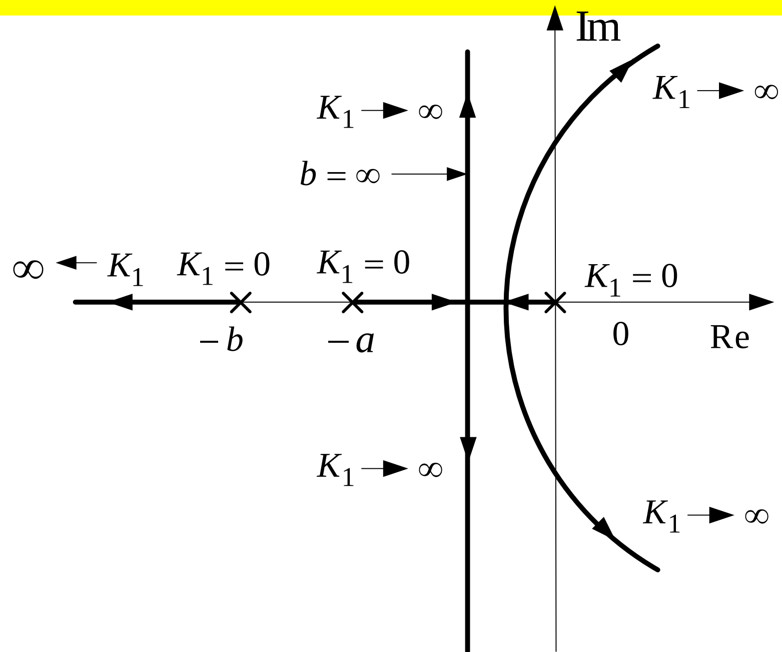
add poles

$$G(s)H(s)=\frac{K_1}{s(s+a)(s+b)} \quad (b>a)$$

4-3 System Analysis Using the Root Locus



(a) $G(s)H(s)=\dfrac{K_1}{s(s+a)}$ $(a>0)$

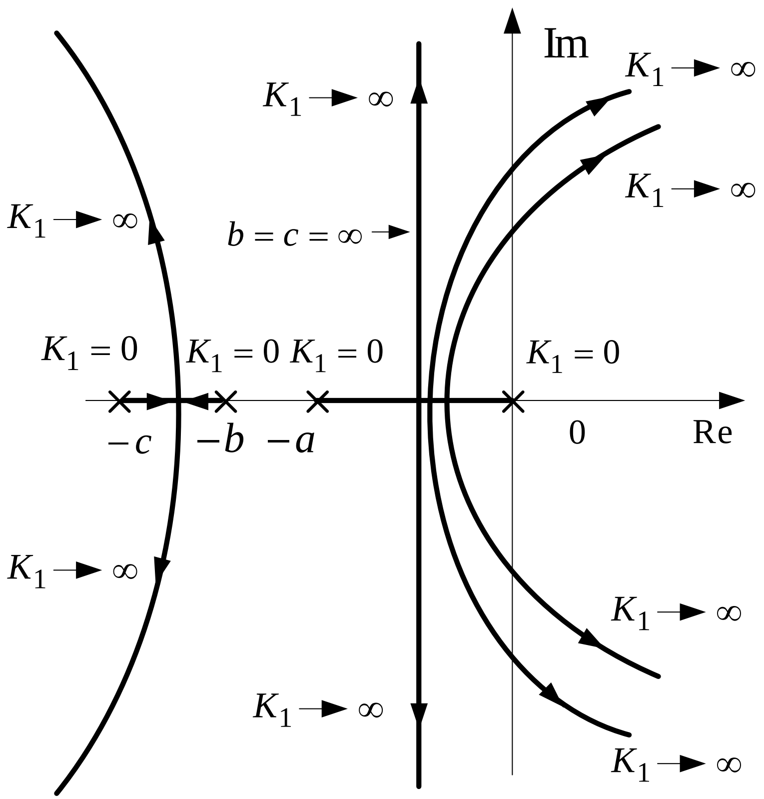


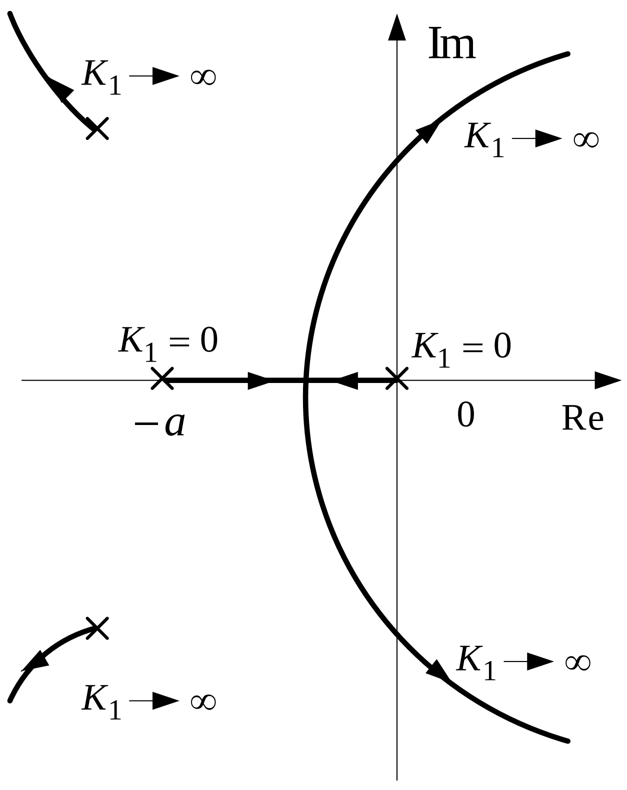
(b) $G(s)H(s)=\dfrac{K_1}{s(s+a)(s+b)}$ $(b>a)$

Increase the pole, the locus change to curve with right way. The Angle of the asymptotes change from $\pm 90^0$ to $\pm 60^0$. Breakaway point move to right. (a) stable, (b) stable in K1 small, K1 big may not stable.

增加极点轨迹向右弯曲，渐近线角度由$\pm 90^0$变为$\pm 60^0$。分离点向右移。（a）稳定，（b）在K_1小时稳定，K_1大可能不稳定。

4-3 System Analysis Using the Root Locus





$$G(s)H(s) = \frac{K_1}{s(s+a)(s+b)(s+c)} \quad (c > b > a)$$

n=4

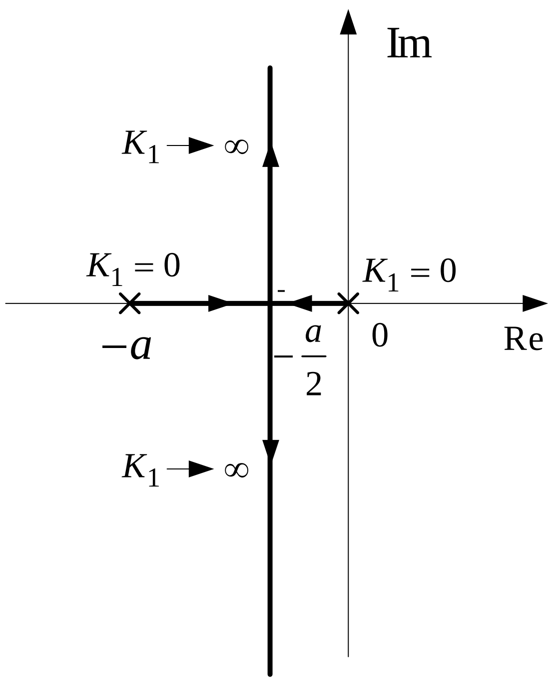
对 $G(s)H(s) = \frac{K_1}{s(s+a)} \quad (a > 0)$ 增加复零点

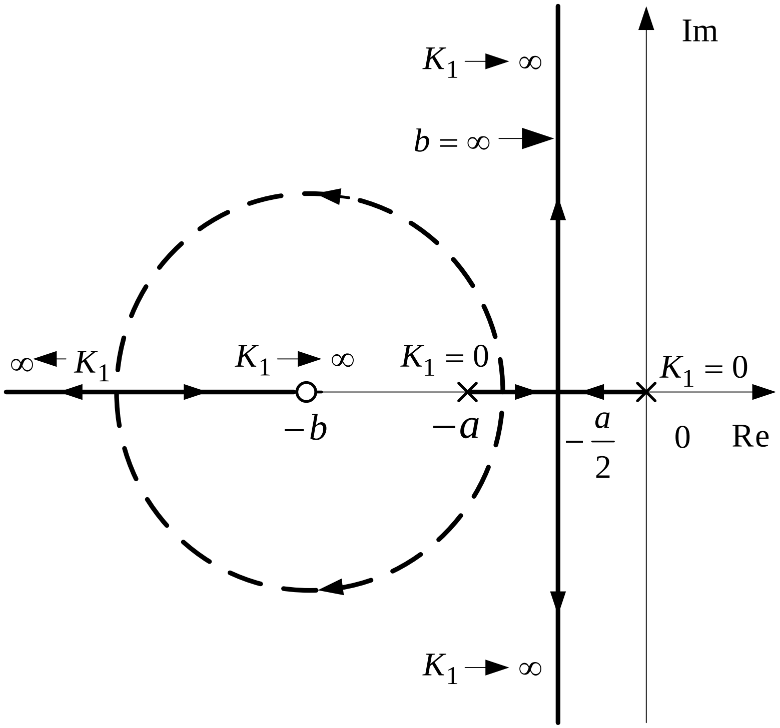
n=2

4-3 System Analysis Using the Root Locus

2. Add zero (with specific system to illustrate)

For G(s) H (s) function, add Zero, can make the root locus move to left way in s plane , the stability of the system is increased.

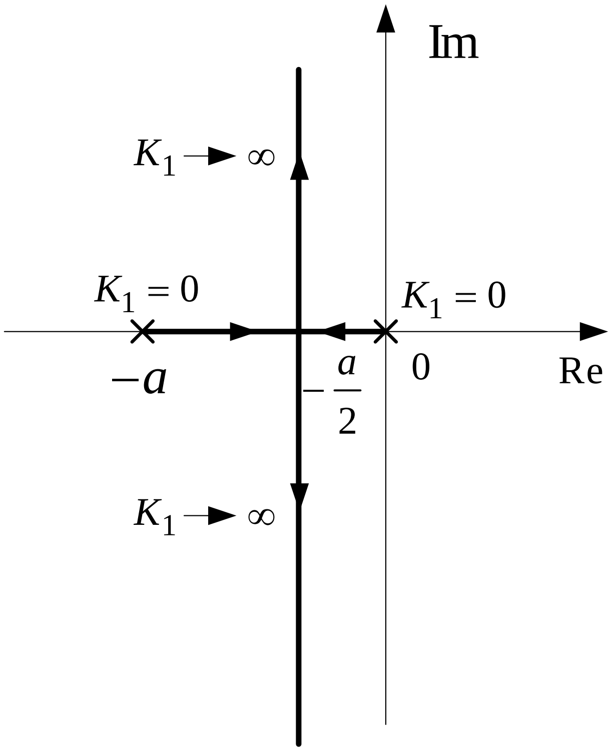


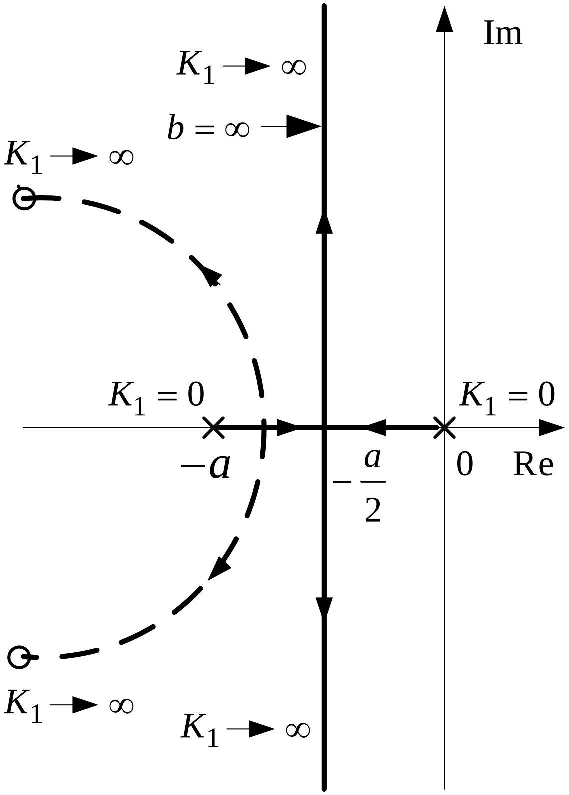


$$G(s)H(s) = \frac{K_1(s+b)}{s(s+a)} \quad (b > a)$$

增加一个零点，根轨迹将向左弯曲形成一个圆

4-3 System Analysis Using the Root Locus





$$G(s)H(s) = \frac{K_1(s+b)}{s(s+a)} \quad (b > a)$$

增加一对轭复数零点后的根轨迹

4-3 System Analysis Using the Root Locus

3. Conclusion

Closed loop zero, poles to the influence of system dynamic performance

- (1) the distribution of close-loop poles determine the type of dynamic response.
- (2) the distribution of closed-loop zero decided the transient response curve form and index.
- (3) real zero will reduce the closed-loop system of damping ratio, make the system speed to fast, overshoots increases, peak time in advance.
- (4) the system dynamic characteristics of the system depends on the close-loop poles.
- (5) away from the imaginary axis of poles (or zero) and dipole can be negligible.

Homework

- Page 226-227-228
  - Ex5.10
  - Ex5.12
  - Ex5.16
  - Ex5.17
  - Deadline:Nov.7.2012

- **Course Project 2: The Root Locus Analysis of the Control System**
- Nov5/11,2012(4 hours)
- Nov12, Chap5