

# Principles of Automatic Control

- Chap 5 The Frequency-Response Design Method  
(5-1 The Definition for Frequency Characteristics)

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# Frequency characteristics (also called frequency response)

- Frequency characteristics of the control system in frequency domain is **a kind of math model**, is **a engineering method** to study the automatic control system.
- The system frequency characteristics can **indirectly reveals system dynamic characteristics and steady-state characteristics**. It can to study or judge rapidly some links parameters to influence on the performance of the system, and points out that the system improved direction.
- Frequency characteristics can **be determined by experiment**. It is very useful for those system which are difficult to build the system dynamic model.

# Introduction(1)

## **The Frequency-Response Design Method Compare with root locus method**

- **Root locus:**
  - **can predict the transient behavior of a closed-loop system with the given open-loop transfer function;**
  - **can determine the stability of the system**
  - **relies on the existence of the open-loop transfer function to draw the locus**
  - **These transfer functions are normally obtained by theoretical modeling**

## Introduction(2)

- **Frequency response**
- **For a controlled object that is too complex model or is unknown (or is uncertain) & uncertain, root locus couldn't be used;**
- **For some circumstances, it would be very dangerous to closed loop before determining its stability**
  - **If the system frequency response could be obtained by experimental modeling, the closed-loop stability would be then determined**
  - **Performance requirements may also be specified the frequency domain**

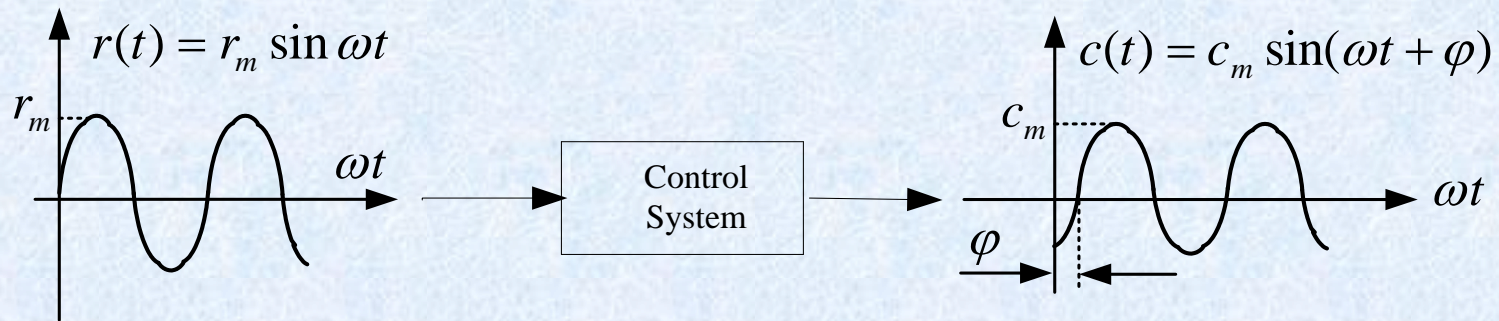


# 5-1 The Definition for Frequency Characteristics

## 1、 Definition :

A linear system's response to sinusoidal inputs – called the system's **Frequency Response**.

We always use  $G(j\omega)$  to express Frequency Response.



That is :  $G(j\omega) = \frac{\dot{c}}{\dot{r}}$  ——— The system output in steady-state

## **5-1 The Definition for Frequency Characteristics**

### **1. The Definition for Frequency Characteristics**

**The stable output response of a system under small harmonic signal with various working frequencies**

- **Frequency response reflects the inner specialty of a system**
- **Spectrum analyzer, such as HP, is one kind of major instruments in experimental modeling**

## 5-1 The Definition for Frequency Characteristics

**Ex: RC circuit**

**input:**  $r(t) = A \sin \omega t$

Capacitance of the complex impedance equivalent for the C:  $\frac{1}{j\omega C}$

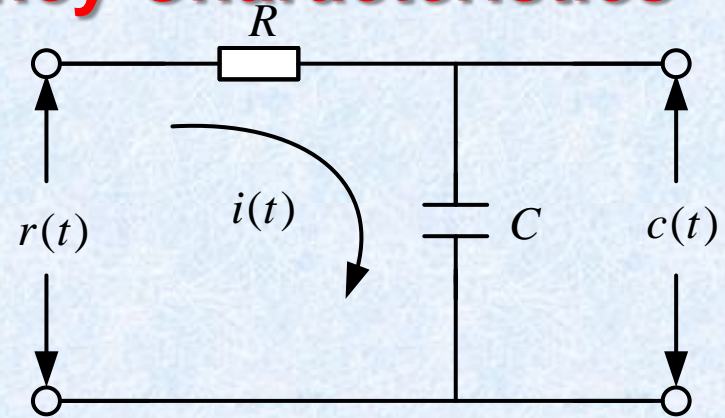
**Output:**

$$\dot{c} = \frac{\dot{I}}{j\omega C} \quad \text{where:} \quad \dot{I} = \frac{\dot{r}}{R + \frac{1}{j\omega C}}$$

The ratio for Circuit output voltage and the input voltage :

$$G(j\omega) = \frac{\dot{c}}{\dot{r}} = \frac{1}{RCj\omega + 1} = \frac{1}{jT\omega + 1}$$

( $RC=T$ ) This is the RC network frequency characteristics



# 5-1 The Definition for Frequency Characteristics

## The properties of the frequency characteristic(1)

1. It is like the transfer function, frequency characteristics is also a kind of **mathematical model**.

It describes the intrinsic characteristics of the system, and have nothing to do with the external factors. When the system structure parameters are given, **the frequency characteristics also fully determined**.

2. frequency characteristics is **a steady-state response** too.

The premise of the system stability is obtained, **for the unstable system cannot directly observed a steady-state response**. In theory, the system dynamic process of steady total can be separated out, and the rules is not dependent on the stability of the system.

Therefore, we can still use frequency characteristics for analysis and study system, including its **stability, dynamic performance, steady performance**, etc ([Chap3 Dynamic Response](#)).



# 5-1 The Definition for Frequency Characteristics

## The properties of the frequency characteristic(2)

3. The system steady output and input should **have the same frequency**. This is caused by energy storage devices of the system.
4. The practical system, or called the real system output as frequency(频率) increases and appear, amplitude(幅值)attenuation and distortion. So, we can see them as a "low pass" filter.
5. The frequency characteristics can be used in some of the nonlinear system analysis.

# 5-1 The Definition for Frequency Characteristics

## How to calculate the Frequency characteristics

### 1. by definition method

That is the differential equation known, and system input function is the Sine function, find out the generation of the steady-state solution, take the steady-state input and output component of the complex, then you will get.

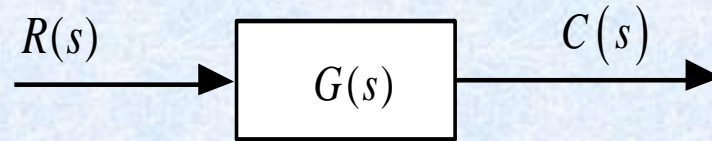
### 2. according to transfer function method

Use  $s = j\omega$  generation into the system transfer function, can get.

### 3. through the experimental method measure directly.

# 5-1 The Definition for Frequency Characteristics

- according to transfer function method :



- transfer function :

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

- Frequency Characteristics : ( $s=j\omega$ )

$$\begin{aligned} G(j\omega) &= \frac{C(j\omega)}{R(j\omega)} = \frac{b_m (j\omega)^m + b_{m-1} (j\omega)^{m-1} + \dots + b_1 (j\omega) + b_m}{a_n (j\omega)^n + a_{n-1} (j\omega)^{n-1} + \dots + a_1 (j\omega) + a_n} \\ &= A(\omega) e^{j\Phi(\omega)} = U(\omega) + jV(\omega) \end{aligned}$$

## 5-1 The Definition for Frequency Characteristics

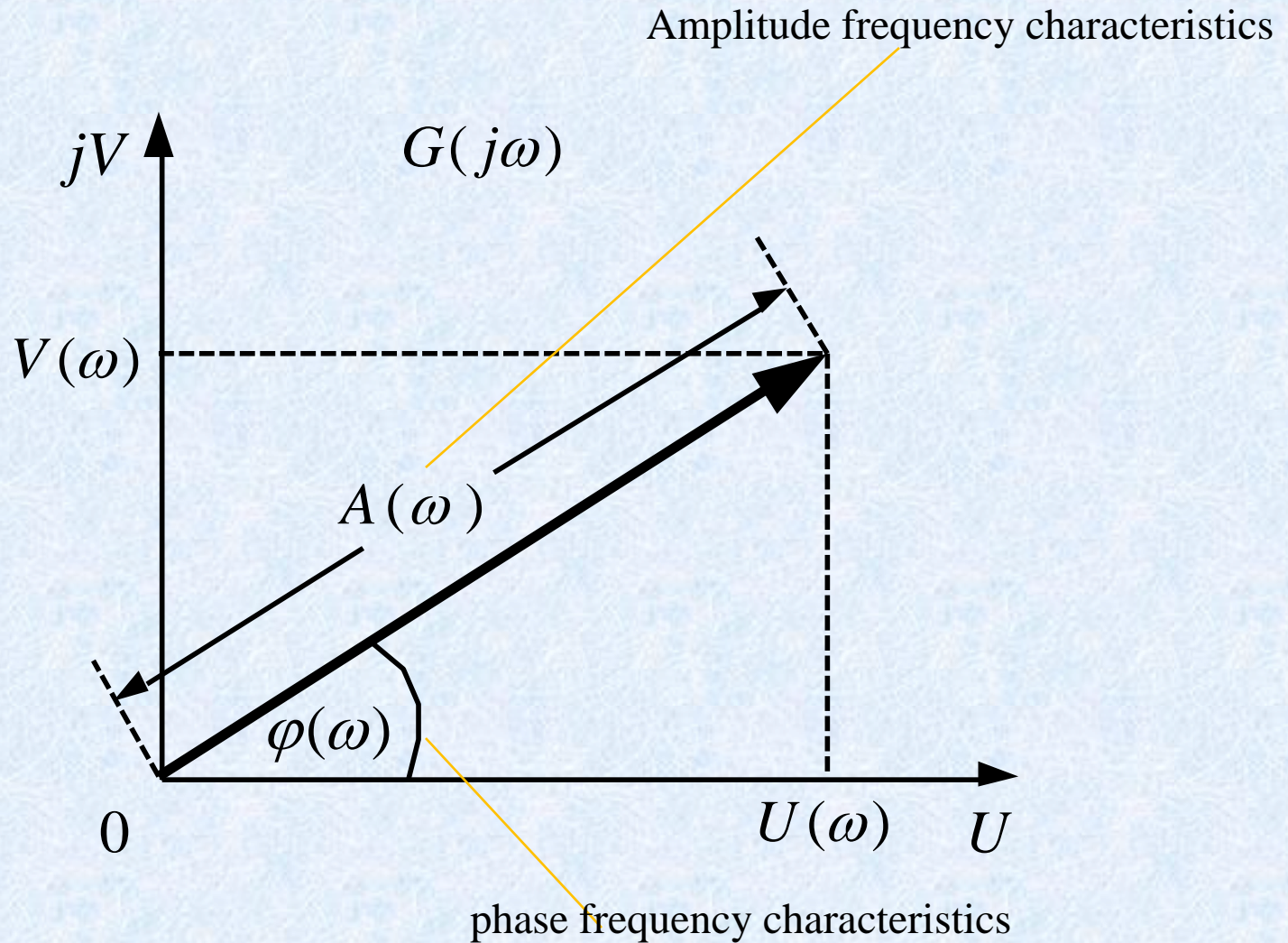
$$G(j\omega) = \frac{C(j\omega)}{R(j\omega)} = A(\omega)e^{j\varphi(\omega)} = U(\omega) + jV(\omega)$$

- A (ω)** —— Amplitude frequency characteristics (幅频特性) ; the magnitude of  $G(j\omega)$ , it is equal to the ratio of the steady state of the input and output component amplitude.
- φ (ω)** —— phase frequency characteristics (相频特性) ; the phase of  $G(j\omega)$ , it is equal to the steady state of Angle output component and the input amount of phase difference.
- U (ω)** —— real frequency characteristics; real part of  $G(j\omega)$  .
- V (ω)** —— virtual frequency characteristics; the imaginary part of  $G(j\omega)$ .

All are the function of  $\omega$ , the relationship between the vector diagram as follows:



# 5-1 The Definition for Frequency Characteristics



# 5-1 The Definition for Frequency Characteristics

## 4、 The three diagram methods for the Frequency characteristics

- **1 . The polar figure-Nyquist Figure** (also called the frequency, amplitude and phase properties figure, or Nyquist figure or chart) **(REQUIRED)**
- **2 .The logarithmic coordinates figure-Bode Figure** (also called Byrd figure, the abbreviation B-chart) **(REQUIRED)**
- **3. The composite coordinates figure-Nichols Figure** ; And are always used for analysis the closed-loop system frequency characteristics. **(KNOW)**

- **Summary for Chap5-1**

**Suppose a stable object  $G(s)$ :**

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{(s - s_1)(s - s_2) \cdots (s - s_n)}$$

$$x(t) = X \sin \omega t \rightarrow X(s) = \frac{X\omega}{s^2 + \omega^2} = \frac{X\omega}{(s + j\omega)(s - j\omega)}$$

$$Y(s) = G(s) \cdot X(s) = \frac{d_1}{(s + j\omega)} + \frac{d_2}{(s - j\omega)} + \frac{c_1}{(s - s_1)} + \frac{c_2}{(s - s_2)} + \cdots + \frac{c_n}{(s - s_n)}$$

$$y_{ss}(t) = d_1 e^{-j\omega t} + d_2 e^{j\omega t}$$

$$d_1 = \left[ G(s) \frac{X\omega}{(s+j\omega)(s-j\omega)} \cdot (s+j\omega) \right]_{s=-j\omega} = -\frac{G(-j\omega)X}{2j}$$

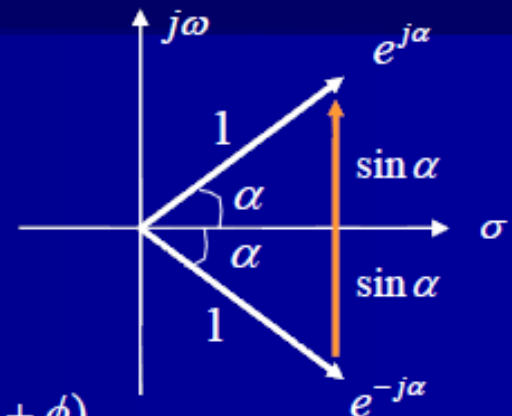
$$d_2 = \left[ G(s) \frac{X\omega}{(s+j\omega)(s-j\omega)} \cdot (s-j\omega) \right]_{s=j\omega} = \frac{G(j\omega)X}{2j}$$

$$G(-j\omega) = |G(-j\omega)| e^{-j\phi} = |G(j\omega)| e^{-j\phi}$$

$$G(j\omega) = |G(j\omega)| e^{j\phi}$$

$$y_{ss}(t) = |G(j\omega)| X \cdot \frac{e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}}{2j} = |G(j\omega)| X \sin(\omega t + \phi)$$

$$\frac{e^{j\alpha} - e^{-j\alpha}}{2j} = \sin \alpha$$



**Thus,**

$$M = |G(j\omega)| \quad \text{Magnitude gain from input to output}$$

$$\phi = \angle G(j\omega) \quad \text{Phase angle difference of output compared with input}$$

$$G(j\omega) = G(s) \Big|_{s=j\omega}$$

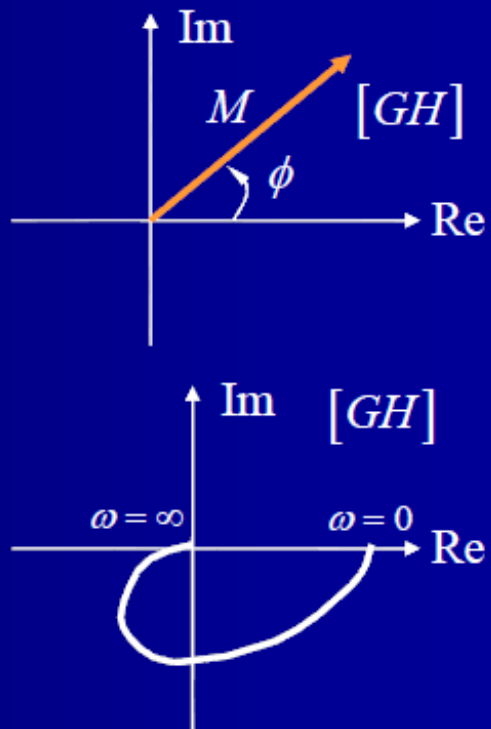
$G(j\omega)$  is on the imaginary axis of the  $S$  plane

**It is an analytical method to achieve the system frequency response**

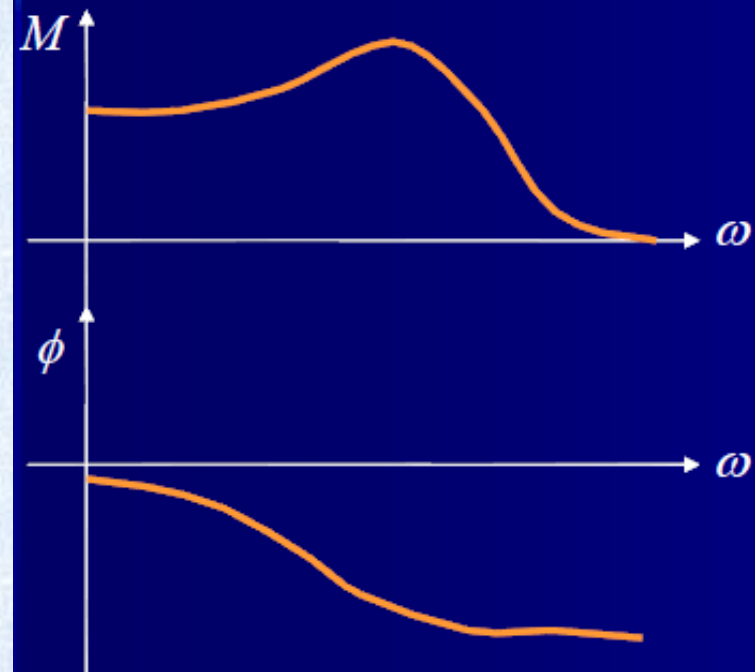


# Frequency response representations

## ■ Nyquist diagram



## ■ Bode diagram



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- Chap 5 The Frequency-Response Design Method  
(5-2 The Typical element analysis with Nyquist diagram )

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## 5-2 The Typical element analysis with Nyquist diagram

### 1. The amplification element ( 放大环节 )

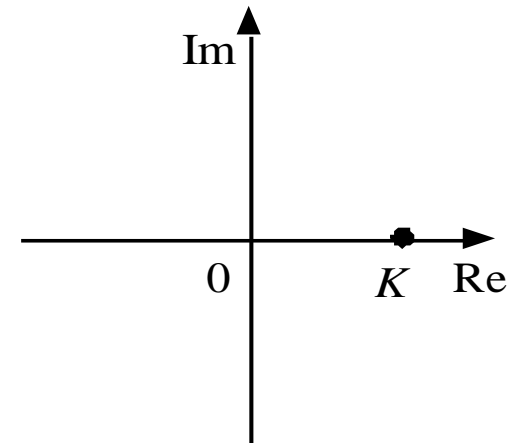
◆  $G(j\omega) = K = U + jV$

$$|G(j\omega)| = \sqrt{U^2 + V^2} = K$$

magnitude

$$\angle G(j\omega) = \tan^{-1} \frac{V}{U} = 0^\circ$$

Phase angle



- ◆ The **Nyquist diagram** of amplification element is a point on the real axis, the distance from the origin O is K.



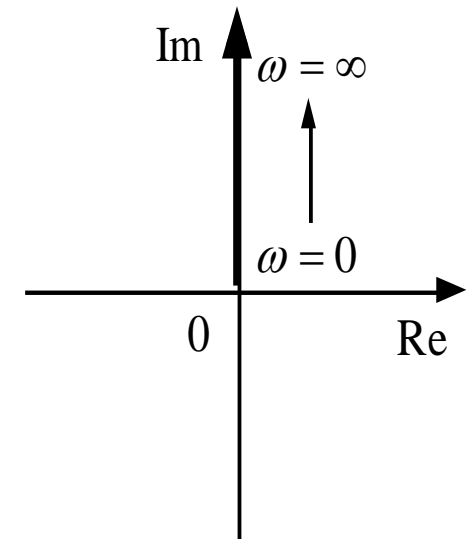
# 5-2 The Typical element analysis with Nyquist diagram

## 2.The Differential element ( 微分环节 )

◆  $G(j\omega) = j\omega$

$|G(j\omega)|$                       magnitude

$\angle G(j\omega) = \tan^{-1} \frac{\omega}{0} = 90^\circ$                       Phase angle



- ◆ The Nyquist diagram of a Differential element is a straight line and reclosing on the imaginary axis.





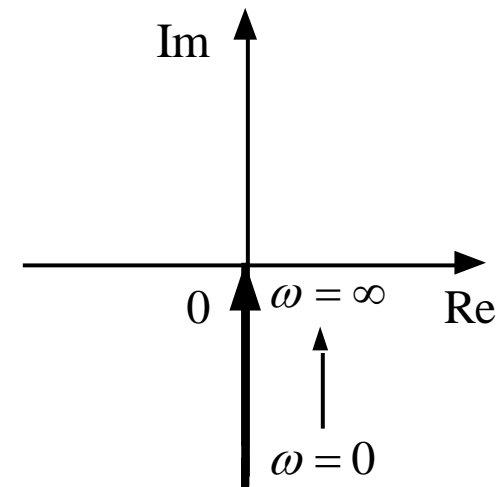
# 5-2 The Typical element analysis with Nyquist diagram

## 3. The integrator element(积分环节)

$$G(j\omega) = \frac{1}{j\omega}$$

$$|G(j\omega)| = \frac{1}{\omega}$$

$$\angle G(j\omega) = \tan^{-1} \frac{-\frac{1}{\omega}}{0} = -90^\circ \quad \text{Phase angle}$$



- ◆ Because of  $\angle G(j\omega) = -90^\circ$ , that is a constant.
- ◆ Therefore, with  $\omega$  decreases, The Nyquist diagram of a integral element is a straight line and reclosing on virtual axis.



# 5-2 The Typical element analysis with Nyquist diagram

## 4. The Inertia element ( 惯性环节 )

$$G(j\omega) = \frac{1}{1 + j\omega T} = \frac{1}{1 + \omega^2 T^2} - j \frac{\omega T}{1 + \omega^2 T^2}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 T^2}} \quad \angle G(j\omega) = \tan^{-1} \frac{-\omega T}{1} = -\omega T$$

◆ Give three Special points , so

$$G(j0) = 1 \angle 0^\circ \quad G\left(j\frac{1}{T}\right) = \frac{1}{\sqrt{2}} \angle 45^\circ \quad G(j\infty) = 0 \angle -90^\circ$$

It can be find, as frequency  $\omega = 0 \rightarrow \infty$ , The amplitude of Inertia element is attenuation, and ultimately tend to 0 gradually. In the absolute value of the displacement is more and more big, but won't end up more than  $90^\circ$ , the polar coordinates the graph is a semicircle.



## 5-2 The Typical element analysis with Nyquist diagram

Set:  $G(j\omega)=U+jV$ , we can proof the Nyquist diagram of this Inertia element is a **semicircle** :

real frequency characteristics  $U = \frac{I}{1 + \omega^2 T^2}$

virtual frequency characteristics  $V = \frac{-\omega T}{1 + \omega^2 T^2}$

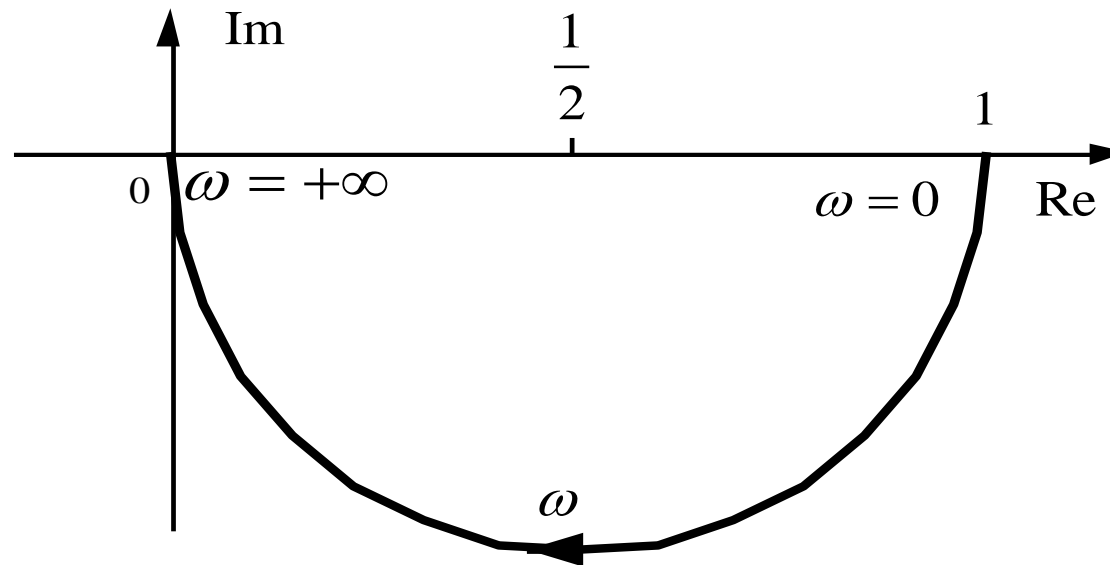
We can get the :  $\frac{V}{U} = -T\omega$  ,so the real frequency characteristics

$$U = \frac{1}{1 + \left(\frac{V}{U}\right)^2} \longrightarrow \left(U - \frac{1}{2}\right)^2 + V^2 = \left(\frac{1}{2}\right)^2$$

It is a Circle equation, circle is  $\left(\frac{1}{2}, 0\right)$   
radius for equation is  $\frac{1}{2}$



## 5-2 The Typical element analysis with Nyquist diagram





# 5-2 The Typical element analysis with Nyquist diagram

## 5. The Oscillation element ( 振荡环节 )

$$\begin{aligned}
 G(j\omega) &= \frac{1}{T^2(j\omega)^2 + 2\zeta Tj\omega + 1} \\
 &= \frac{1 - T^2\omega^2}{(1 - T^2\omega^2)^2 + (2\zeta T\omega)^2} - j \frac{2\zeta T\omega}{(1 - T^2\omega^2)^2 + (2\zeta T\omega)^2} \\
 |G(j\omega)| &= \frac{1}{\sqrt{(1 - T^2\omega^2)^2 + (2\zeta T\omega)^2}} \\
 \angle G(j\omega) &= \operatorname{tg}^{-1} \frac{-2\zeta T\omega}{1 - T^2\omega^2} \quad G(j0) = 1 \angle 0^\circ
 \end{aligned}$$

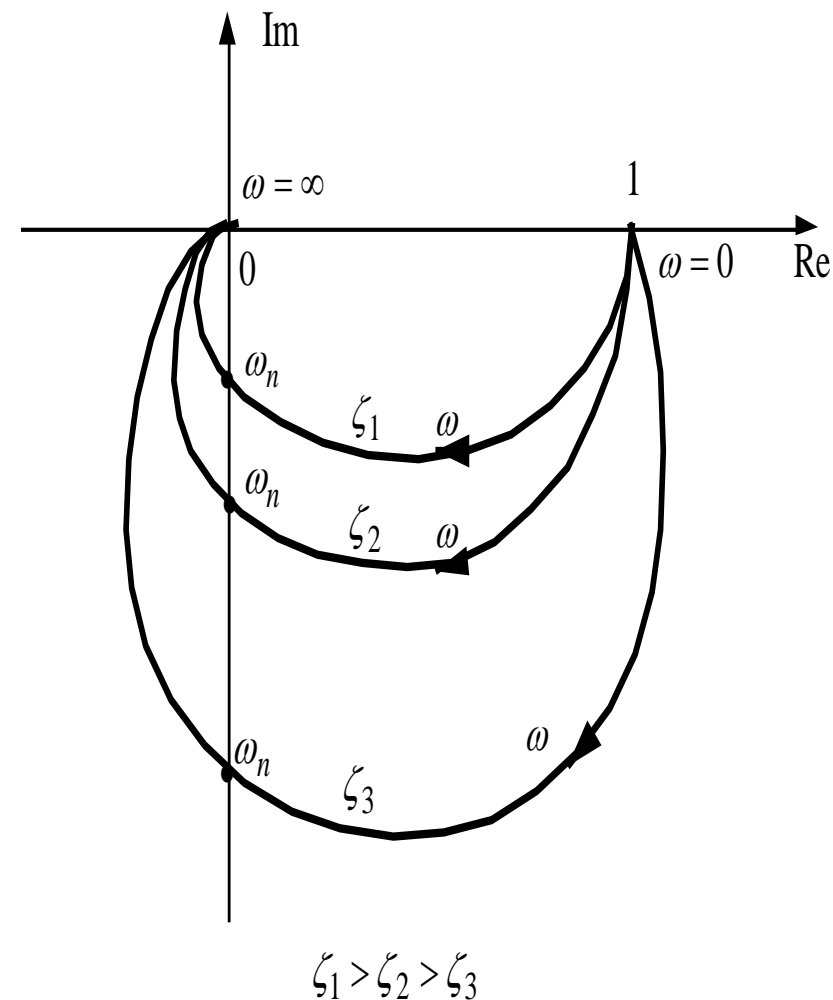
◆ so , when  $\omega=0$  , and  $\omega=\infty$  ,

$$G\left(j\frac{1}{T}\right) = \frac{1}{2\zeta} \angle -90^\circ \quad G(j\omega) = 0 \angle -180^\circ$$



# 5-2 The Typical element analysis with Nyquist diagram

- ◆ The phase of Polar coordinates from 0 to  $-180$  change. The cross point of frequency characteristics and the imaginary axis is the natural oscillation frequency  $\omega_n$ .
- ◆ the smaller the corresponding  $\zeta$  of omega, the greater the amplitude. That frequency characteristics are relevant to  $\omega$  and  $\zeta$ .
- ◆ When  $\zeta$  small enough, there will be a peak, the value is called resonance peak, this time the corresponding frequency called the resonant frequency  $\omega_r$ .



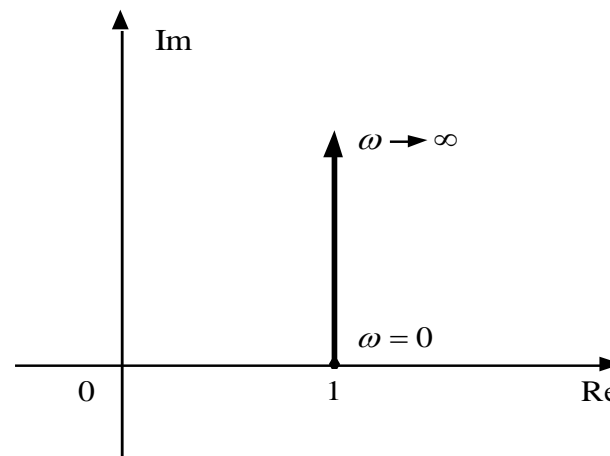
# 5-2 The Typical element analysis with Nyquist diagram

## 6. One order Differential element ( 一阶微分环节 )

$$G(j\omega) = 1 + j\omega T \quad |G(j\omega)| = \sqrt{1 + \omega^2 T^2}$$

$$\angle G(j\omega) = \tan^{-1} \omega T$$

When  $\omega: 0 \rightarrow \infty$ , Phase angle:  $0^\circ \rightarrow +90^\circ$ , the Nyquist diagram is a line, which pass the point ( 1 , 0 ), And parallel to positive virtual axis.



# 5-2 The Typical element analysis with Nyquist diagram

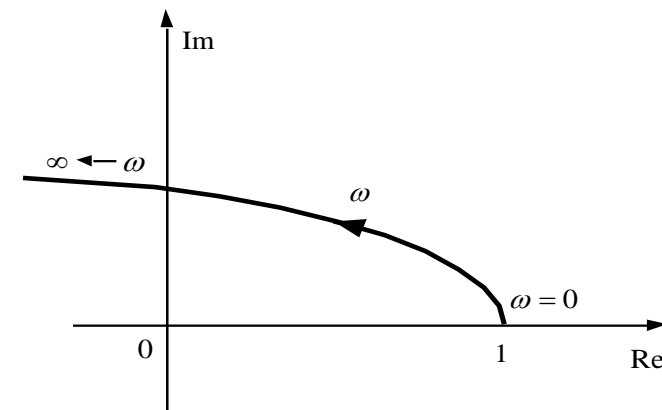
## 7. two order Differential element (二阶微分环节)

$$G(j\omega) = T^2(j\omega)^2 + 2\zeta T(j\omega) + 1 = (1 - T^2\omega^2) + j2\zeta T\omega$$

$$|G(j\omega)| = \sqrt{(1 - T^2\omega^2)^2 + 4\zeta^2 T^2\omega^2}$$

$$\angle G(j\omega) = \operatorname{tg}^{-1} \frac{2\zeta T\omega}{1 - T^2\omega^2}$$

- ◆ With the increase of  $\omega$ ,  $G(j\omega)$  imaginary part is **monotonous increasing** (单调增), and is **real part** drab descending from 1.



# 5-2 The Typical element analysis with Nyquist diagram

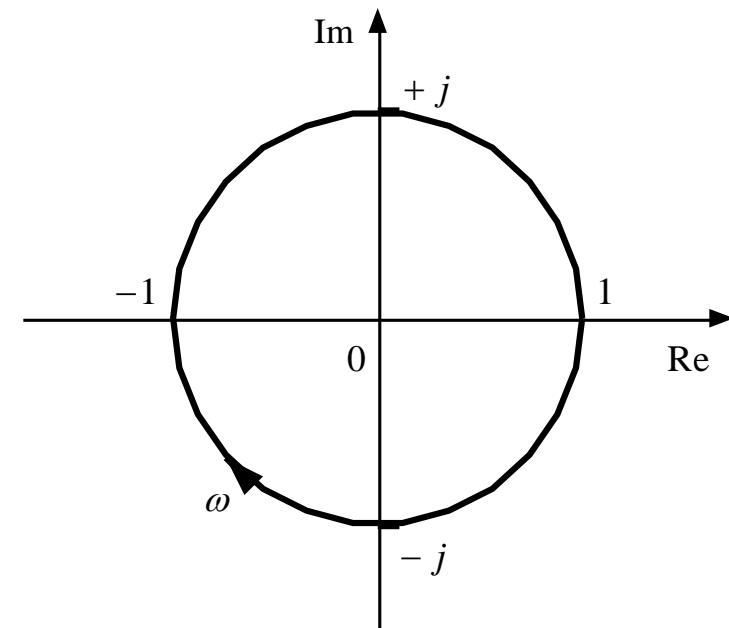
## 8. The Delay element ( 延迟环节 )

$$G(j\omega) = e^{-j\omega T}$$

$$|G(j\omega)| = 1$$

$$\angle G(j\omega) = -\omega T$$

- ◆ The Delay element amplitude frequency characteristics is the constant, which has nothing to do with  $\omega$ , its value is 1. And phase frequency characteristics is linearly with  $\omega$ . So the polar figure is **an unit circle**.





# 5-2 The Typical element analysis with Nyquist diagram

## 2.The open loop system amplitude and phase frequency characteristics

To drawing **Nyquist diagram** for **open loop** system frequency characteristics, it is required to put the system contains every element of the corresponding frequency, **amplitude is multiplied** , **phase is added together**.

**Ex :** To drawing Nyquist diagram for open loop system :

$$G(j\omega) = \frac{e^{-j\omega T}}{1 + j\omega T}$$

**Solution :**  $G(j\omega)$  can be waited as :  $G(j\omega) = e^{-j\omega T} \cdot \frac{1}{1 + j\omega T}$



## 5-2 The Typical element analysis with Nyquist diagram

- ◆ Its Magnitude and Phase angle are calculated as

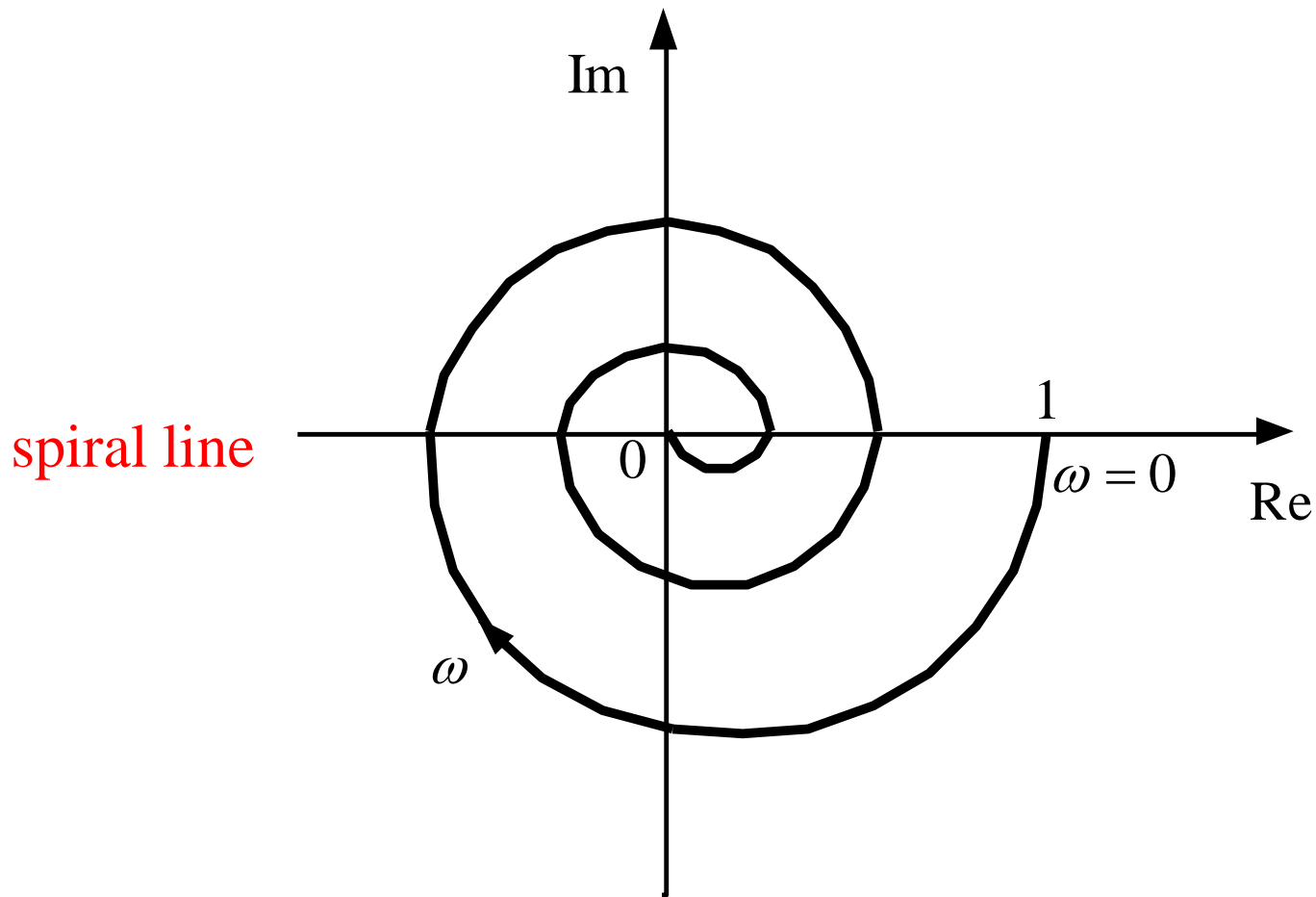
$$|G(j\omega)| = \left| e^{-j\omega T} \right| \left| \frac{1}{1 + j\omega T} \right| = \frac{1}{\sqrt{1 + \omega^2 T^2}}$$

$$\angle G(j\omega) = \angle e^{-j\omega T} + \angle \frac{1}{1 + j\omega T} = -\omega T - \tan^{-1} \omega T$$

Because amplitude is from 1 began to reduce  
phase Angle is drab reduced, so the transfer  
function of polar figure(Nyquist diagram ) is a  
**spiral line.**



## 5-2 The Typical element analysis with Nyquist diagram



## 5-2 The Typical element analysis with Nyquist diagram

◆ Set the open loop transfer function is:

$$G(j\omega)H(j\omega) = \frac{K(1 + j\omega T_a)(1 + j\omega T_b) \cdots \cdots}{(j\omega)^N (1 + j\omega T_1)(1 + j\omega T_2) \cdots \cdots}$$

**System model: a basis in open-loop transfer function system integrator of how much to the classification method of system:**

**1.0 type system ( $N = 0$ )**

**2.Type I system ( $N = 1$ )**

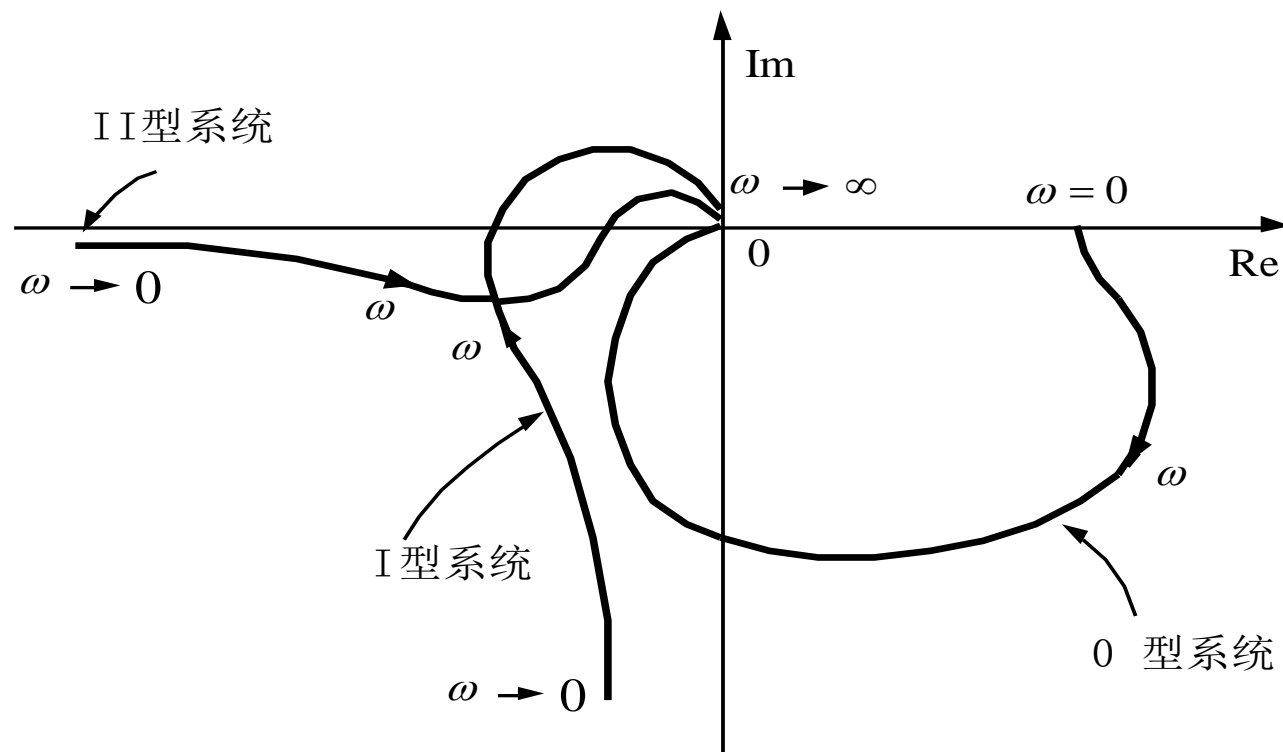
**3.Type II system ( $N = 2$ )**

.....



## 5-2 The Typical element analysis with Nyquist diagram

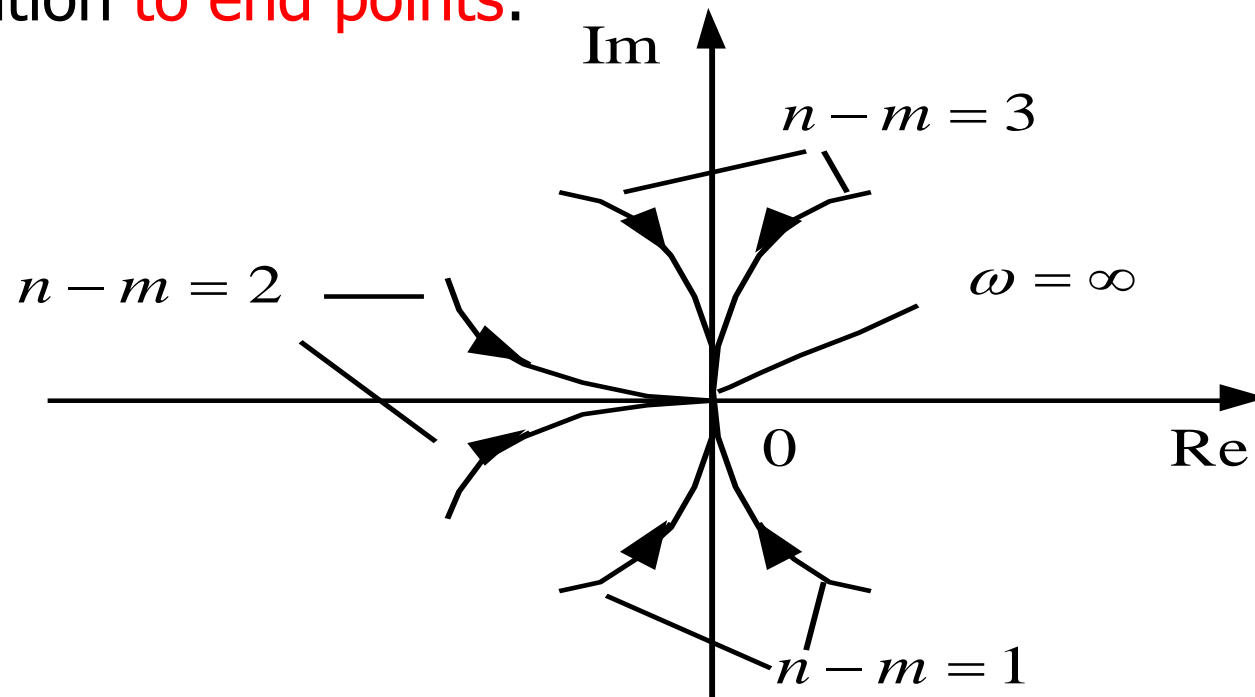
- ◆ The shape of the polar coordinates system graph has relevant to system model, the general situation as follows (note starting point) :





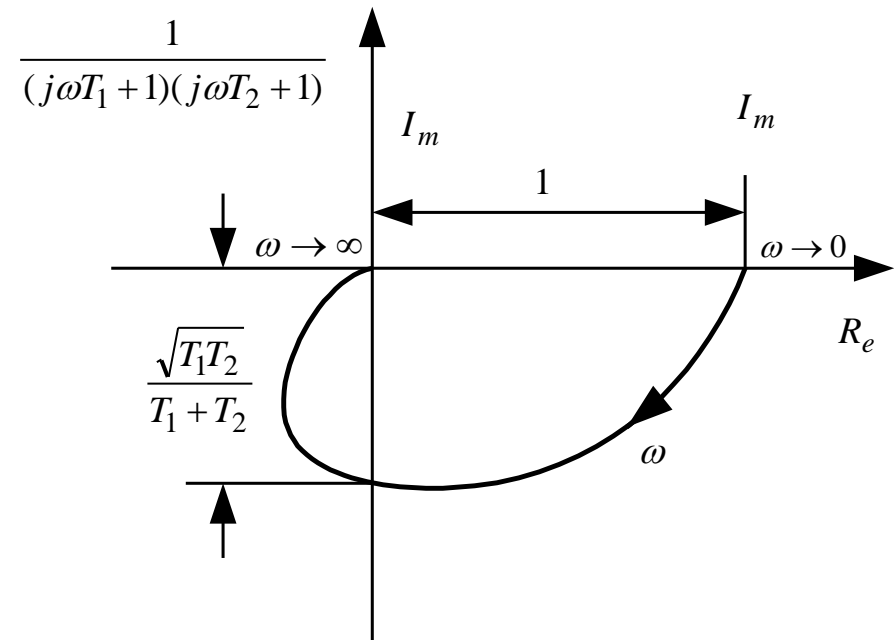
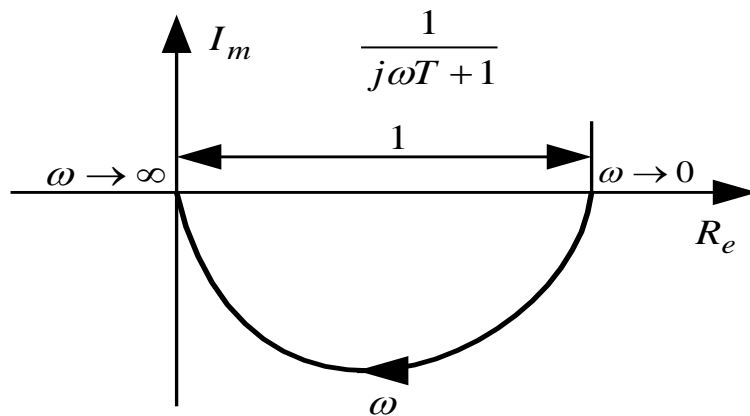
# 5-2 The Typical element analysis with Nyquist diagram

Pay attention to end points:

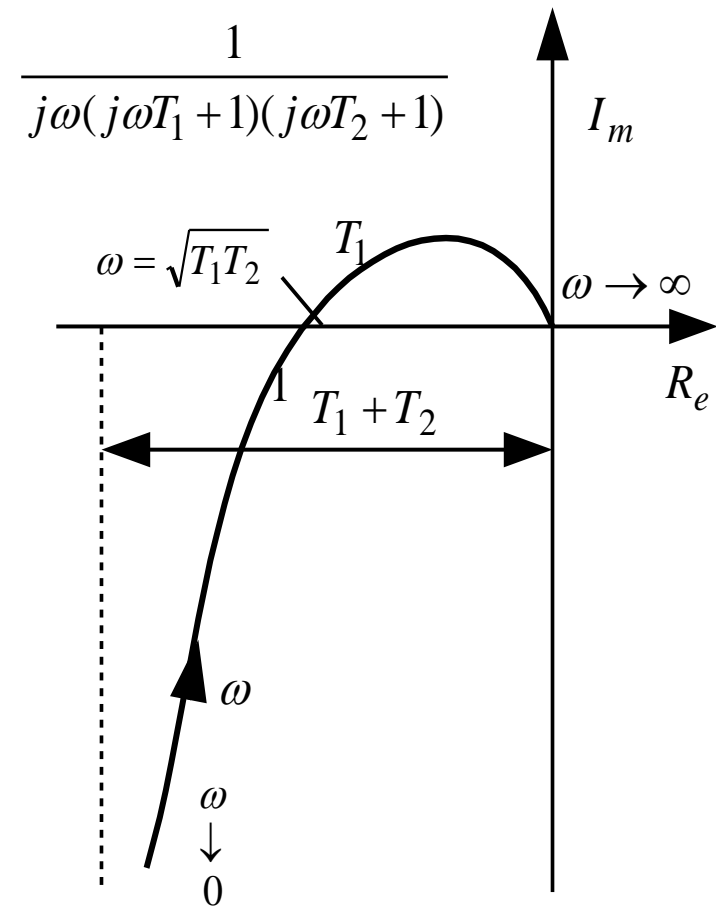
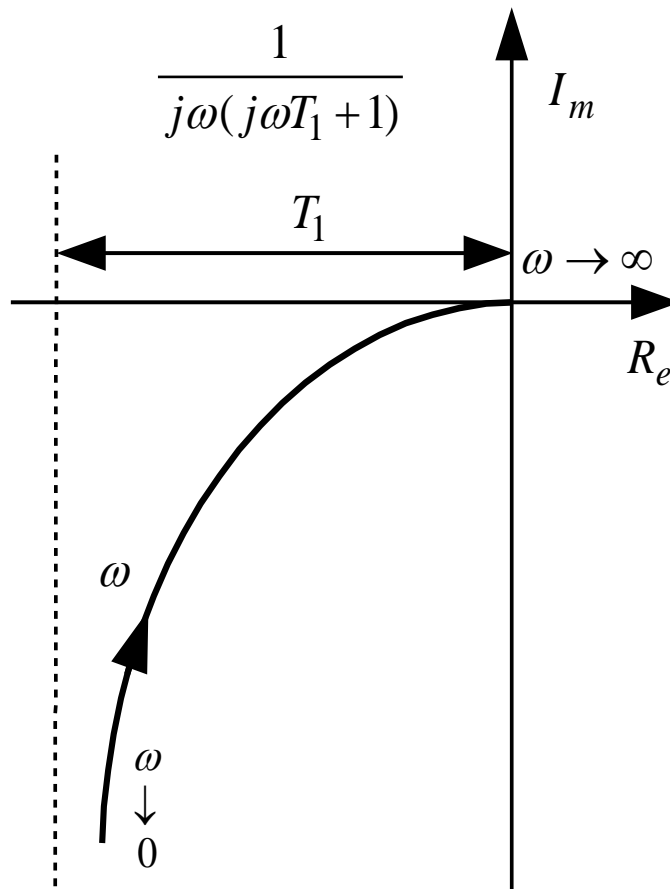


$$G(j\omega) = \frac{b_m(j\omega)^m + \dots}{a_n(j\omega)^n + \dots}$$

# 5-2 The Typical element analysis with Nyquist diagram



# 5-2 The Typical element analysis with Nyquist diagram



## 5-2 The Typical element analysis with Nyquist diagram

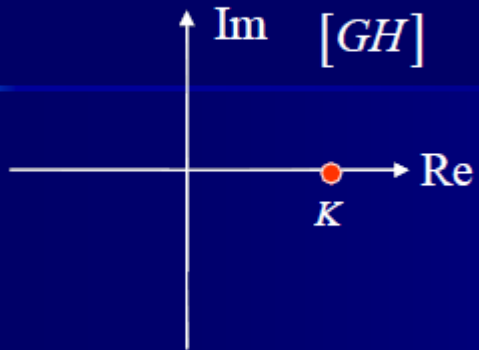
### Conclusion:

- ◆ 1.0 type system ( $N = 0$ ) : polar figure begins in the real axis is the limited point and terminated in origin.
- ◆ 2. Type I system ( $N = 1$ ) : because it has an integral link, so low frequency, polar figure is a gradually to virtual axis parallel straight line and. When  $\omega = \infty$ , amplitude is zero, curve convergence of origin and a tangent coordinates.
- ◆ 3. Type II system ( $N = 2$ ) : low frequency place, polar figure is a gradually to negative real axis of straight. In the  $\omega = \infty$  place of the value is zero, and the curve in a tangent coordinates.

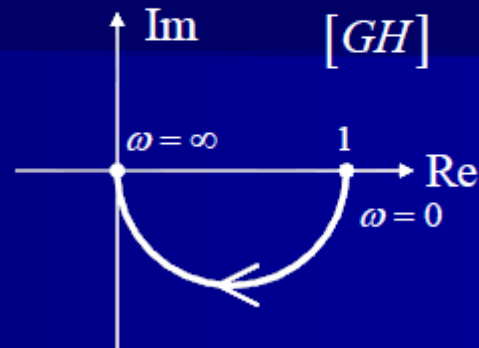


# Nyquist diagram

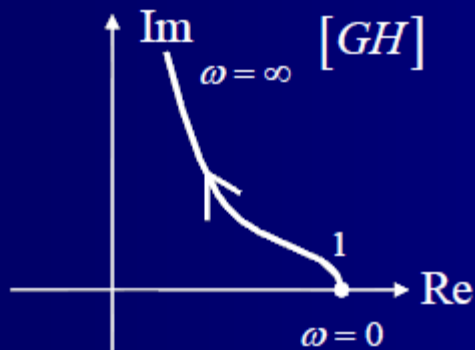
1.  $GH(s) = K$



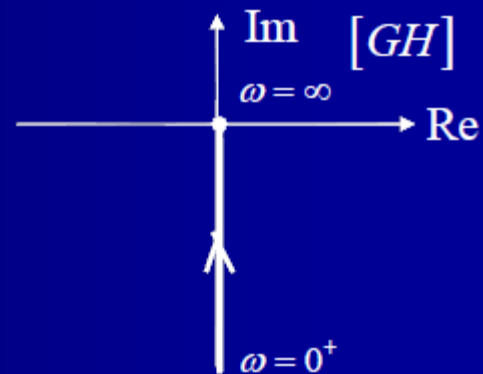
2.  $GH(s) = \frac{1}{1+Ts}$



3.  $GH(s) = 1+Ts$

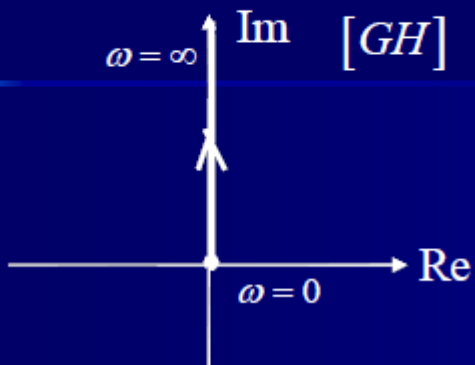


4.  $GH(s) = \frac{1}{s}$





5.  $GH(s) = s$

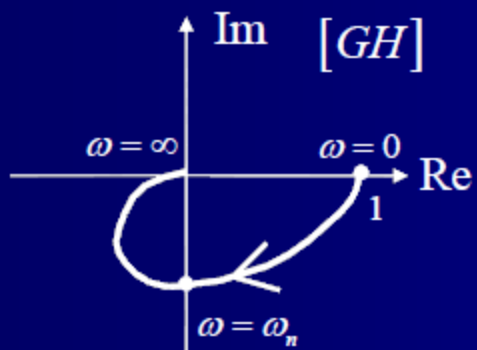


6.  $GH(s) = \frac{1}{s^n}$

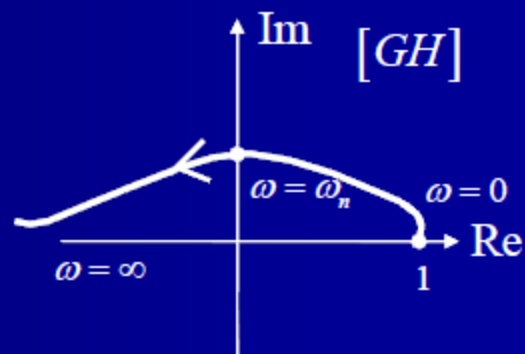
$$M = \frac{1}{\omega^n}$$

$$\phi = -90n$$

7.  $GH(s) = \frac{1}{1 + (2\xi / \omega_n)s + s^2 / \omega_n^2}$



8.  $GH(s) = 1 + (2\xi / \omega_n)s + s^2 / \omega_n^2$



Sketch the Nyquist diagram with a few important points accurately on the  $[GH]$  plane

Open-loop system

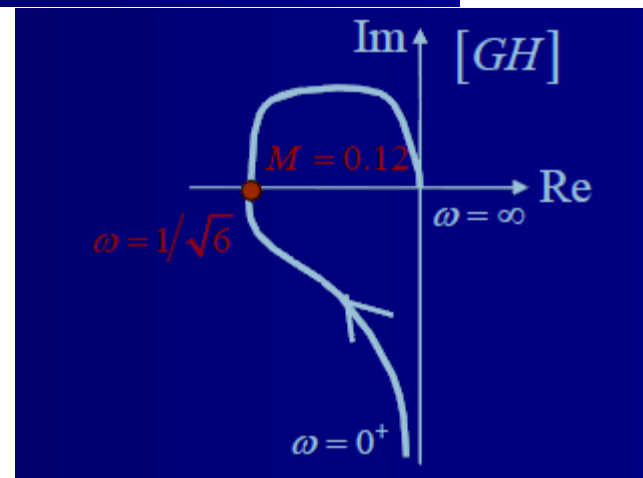
$$GH(s) = \frac{0.1}{s(1+2s)(1+3s)}$$

$$M = \frac{0.1}{\omega \sqrt{1+4\omega^2} \sqrt{1+9\omega^2}}$$

$$\phi = -90 - \tan^{-1} 2\omega - \tan^{-1} 3\omega$$

$$\omega \rightarrow 0: M \rightarrow \infty, \phi \rightarrow -90$$

$$\omega \rightarrow \infty: M \rightarrow 0, \phi \rightarrow -270$$



Cross magnitude and frequency with negative axis is an important point

$$-90 - \tan^{-1} 2\omega - \tan^{-1} 3\omega = -180$$

$$\tan^{-1} 2\omega + \tan^{-1} 3\omega = 90$$

$$\frac{2\omega + 3\omega}{1 - 2\omega \cdot 3\omega} = \infty \Rightarrow \omega = \frac{1}{\sqrt{6}}$$

$$M = \frac{0.1}{\frac{1}{\sqrt{6}} \sqrt{1+4 \cdot \frac{1}{6}} \sqrt{1+9 \cdot \frac{1}{6}}} = 0.12$$



# Principles of Automatic Control

## - Chap 5 The Frequency-Response Design Method (5-3 Bode Diagram )

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# 5-3 Bode Diagram

## ◆ Bode Diagram

- ◆ **H.W. Bode** developed for hand plotting at Bell Laboratories between 1932-1942.
- ◆ Bode Diagram allows plotting the frequency response is **quickly** and yet **sufficiently accurate** for control systems design.
- ◆ The idea in Bode's method is to plot **magnitude curves using a logarithmic scale** and **phase curves using a linear scale**.





# 5-3 Bode Diagram

## ◆ Bode Diagram

### 1. **magnitude curves**(幅频特性)

Vertical –coordinate :  $20\lg|G(j\omega)|$  (dB) , it is a linear scale

Horizontal –coordinate :  $\lg\omega$  , it is a non-linear scale.

### 2. **phase curves** (相频特性)

Vertical –coordinate:  $\varphi(\omega)$  , degree , it is a linear scale too ;

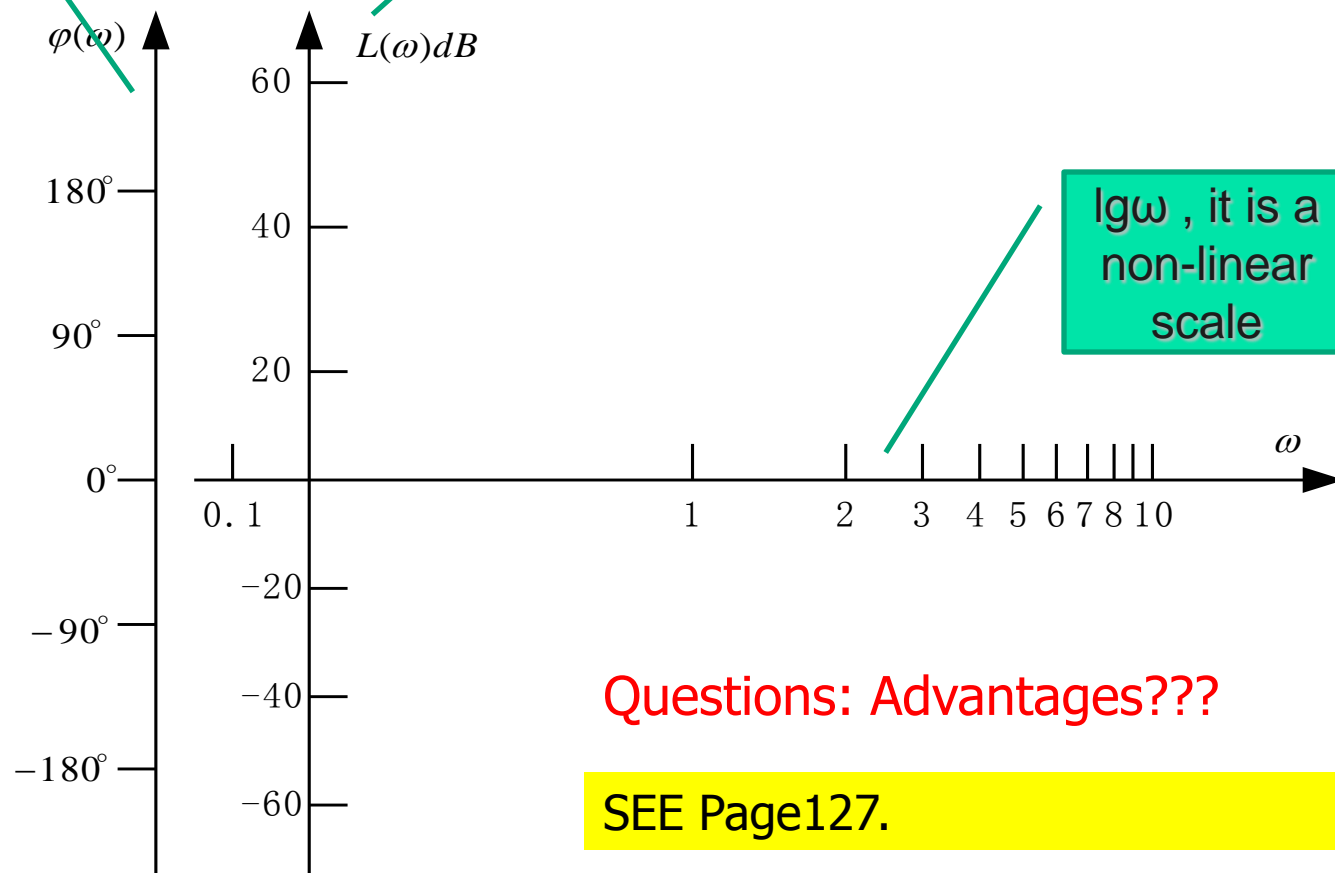
Horizontal coordinate:  $\lg\omega$  , it is a non-linear scale .



# 5-3 Bode Diagram

phase curves  $\varphi(\omega)$ ,  
degree, it is a linear  
scale too

magnitude curves:  
 $20\lg|G(j\omega)|$  (dB), it is a  
linear scale



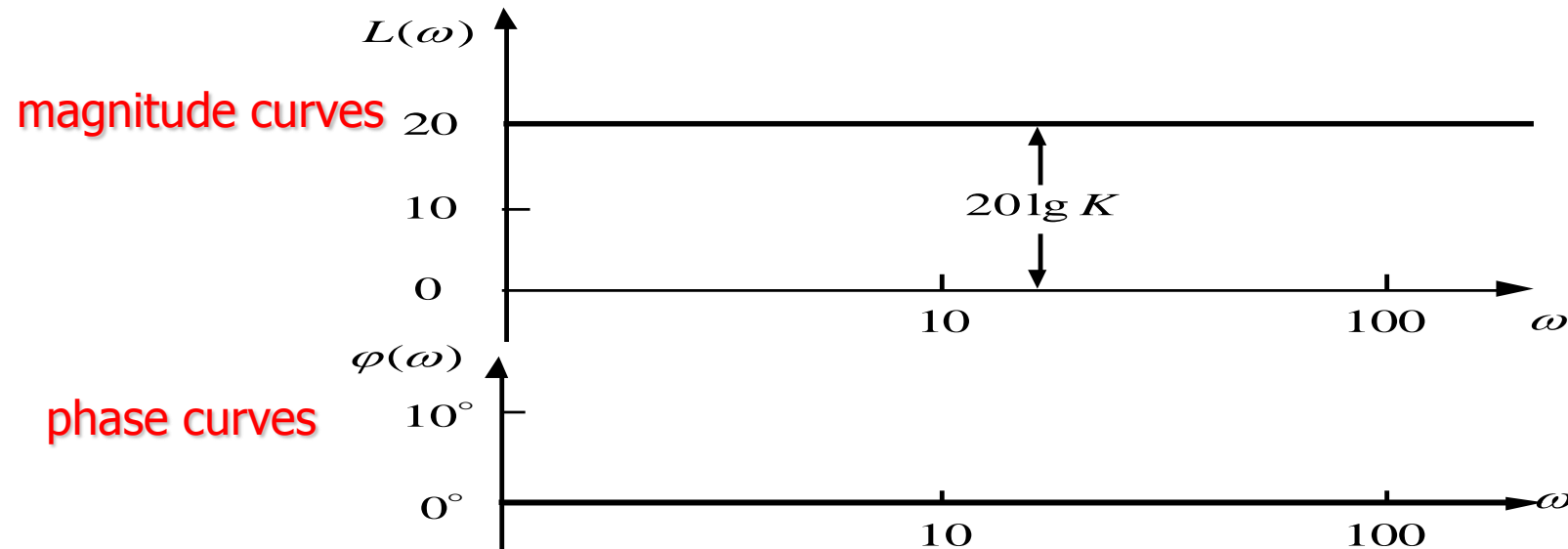
Questions: Advantages???

SEE Page127.



# 5-3-1 The Typical element Bode Diagram

## 1. Amplification element $G(j\omega)=K$



**magnitude curves:** it is a line with value  $20\lg K$  db , and parallel to the horizontal axis.

**phase curves:** it is a line coincidence with Horizontal coordinate.

**when  $K>1$  ,  $20\lg K>0\text{dB}$  ; when  $K<1$  ,  $20\lg K<0\text{dB}$**



# 5-3-1 The Typical element Bode Diagram

**2. The integrator element**  $G(j\omega) = \frac{1}{j\omega}$

$$L(\omega) = 20\lg|G(j\omega)| = 20\lg\left|\frac{1}{j\omega}\right| = -20\lg\omega \quad (\text{dB})$$

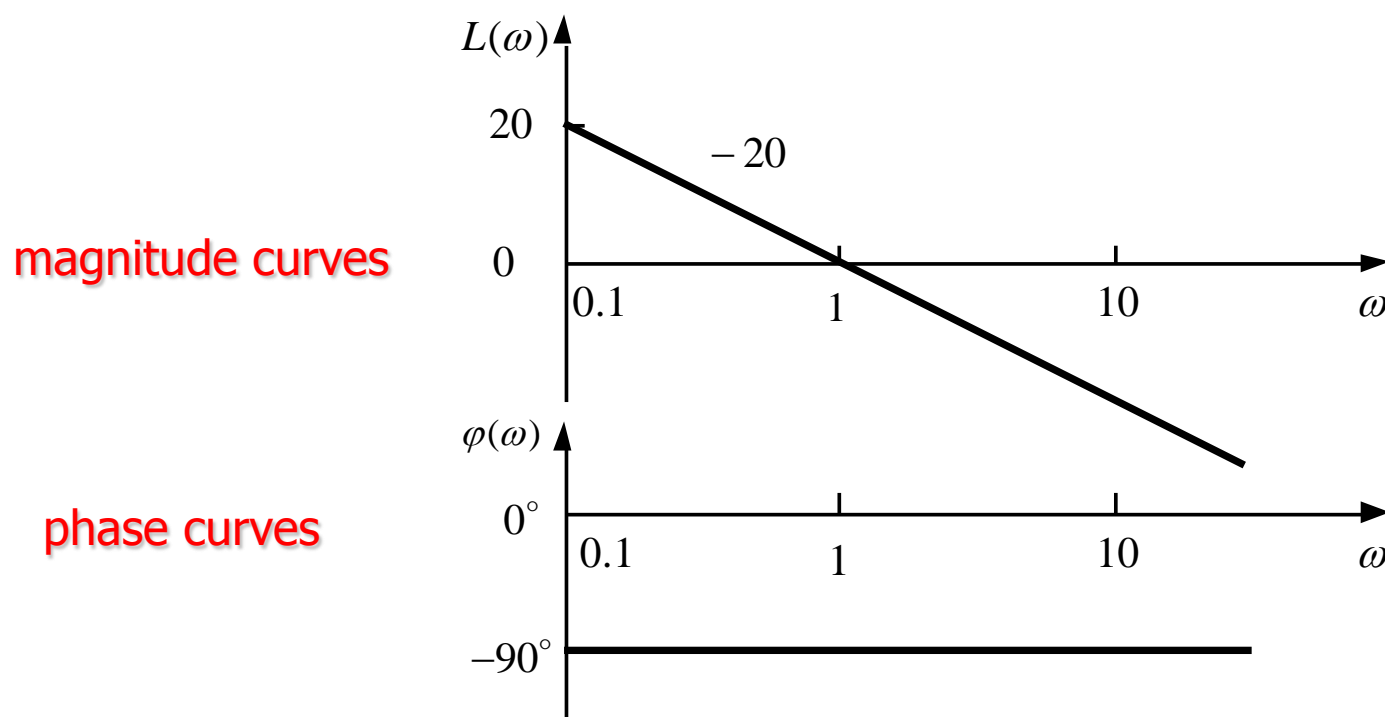
When  $\omega=1$   $L(\omega) = -20\lg 1 = 0 \quad \text{dB}$

When  $\omega=10$   $L(\omega) = -20\lg 10 = -20 \quad \text{dB}$

When  $\omega$  increase 10 time ,  $L(\omega)$  will decrease 20dB. We also note as: -20dB/dec or -20.



# 5-3-1 The Typical element Bode Diagram



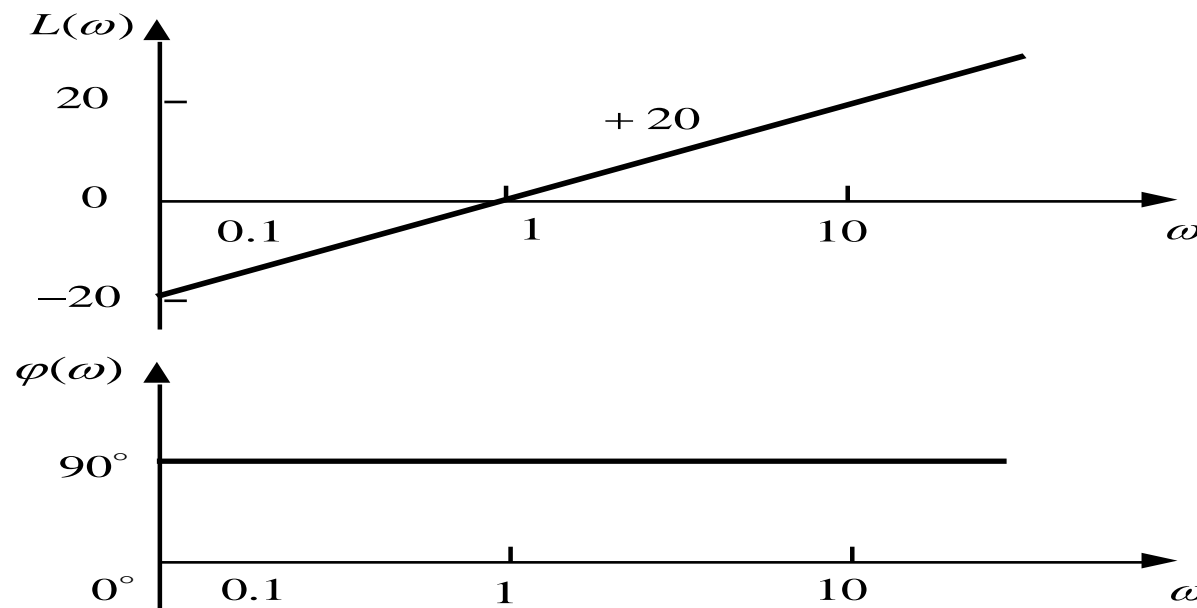
The integrator element's **magnitude curves** is a line which has cross point ( $\omega=1$ ) with Horizontal coordinate , its slope is -20 ; **phase curves** is a line with value  $-90^\circ$  line coincidence with Horizontal coordinate. It is non-realship with  $\omega$  .



# 5-3-1 The Typical element Bode Diagram

## 3 The Differential element $G(j\omega) = j\omega$

Differential element is the integral element of the reciprocal, their slope and phase displacement is just **a minus**.



# 5-3-1 The Typical element Bode Diagram

## 4. The Inertia element

$$G(j\omega) = \frac{1}{1 + j\omega T} = \frac{1}{1 + \omega^2 T^2} - j \frac{\omega T}{1 + \omega^2 T^2}$$

### ◆ magnitude curves

$$20\lg \left| \frac{1}{1 + j\omega T} \right| = 20\lg \frac{1}{\sqrt{1 + \omega^2 T^2}} = -20\lg \sqrt{1 + \omega^2 T^2}$$

when  $\omega \ll \frac{1}{T}$  ( low frequency ) :

It is **approximately** that inertial element at the logarithm of low frequency band amplitude frequency characteristics is straight line which collocated with the horizontal axis.

$$-20\lg \sqrt{1 + \omega^2 T^2} = -20\lg 1 = 0 \text{ (dB)}$$





# 5-3-1 The Typical element Bode Diagram

When  $\omega \gg \frac{1}{T}$  ( High frequency ) :

amplitude frequency characteristics :

$$-20\lg\sqrt{1+\omega^2 T^2} = -20\lg\omega T \text{ (dB)}$$

it is line equation and it will cross the with Horizontal coordinate at  $\omega = \frac{1}{T}$  , its slope is -20 dB/dec.

**Conclusion :** The amplitude frequency characteristics of Inertia element is expressed as two asymptote (渐近线) which are cross at  $\omega = \frac{1}{T}$  :

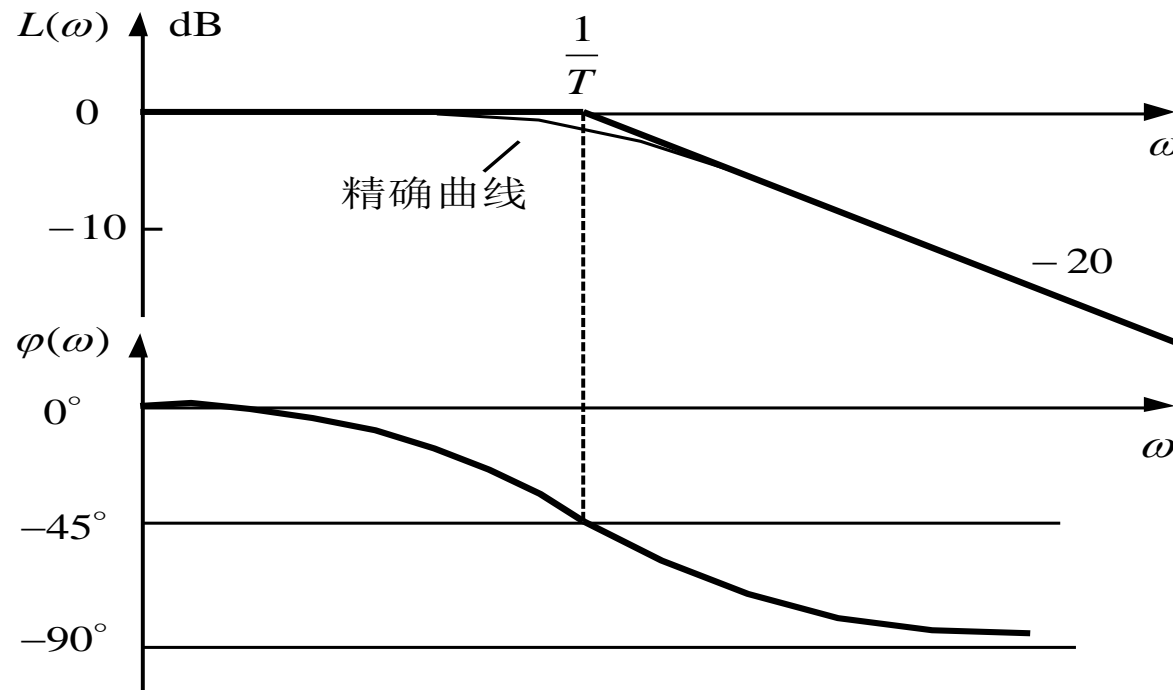
when  $\omega \ll \frac{1}{T}$  , a line with 0db ;

when  $\omega \gg \frac{1}{T}$  , a line with -20dB/dec.



# 5-3-1 The Typical element Bode Diagram

The frequency of two asymptote cross together is:  
 $\omega = \frac{1}{T}$  , which is so-called the **turn frequency** or **cross frequency** .



# 5-3-1 The Typical element Bode Diagram

- ◆ The phase frequency characteristics of Inertia element is:

$$\varphi(\omega) = -\tan^{-1} \omega T$$

when  $\omega=0$ ,  $\varphi(\omega)=0^\circ$ , when  $\omega = \frac{1}{T}$ ,  $\varphi(\omega)=-45^\circ$  ; when  $\omega \rightarrow \infty$ ,  $\varphi(\omega) \rightarrow -90^\circ$ .

The error on the **asymptote** for the **amplitude frequency characteristics** can be calculated. The maximum error happened at the **turn frequency**  $\omega = \frac{1}{T}$ .

$$\text{Error(max)} \approx -20\lg\sqrt{1+1} = -10\lg 2 = -3.01(\text{dB})$$

**Analysis shows that** the inertial element with low pass characteristics of low-frequency input can accurately retrieval, and the high frequency to enter attenuation, and produce phase after late. Therefore, it can only be used repetition or slow change signal.



# 5-3-1 The Typical element Bode Diagram

## 5. One order Differential element

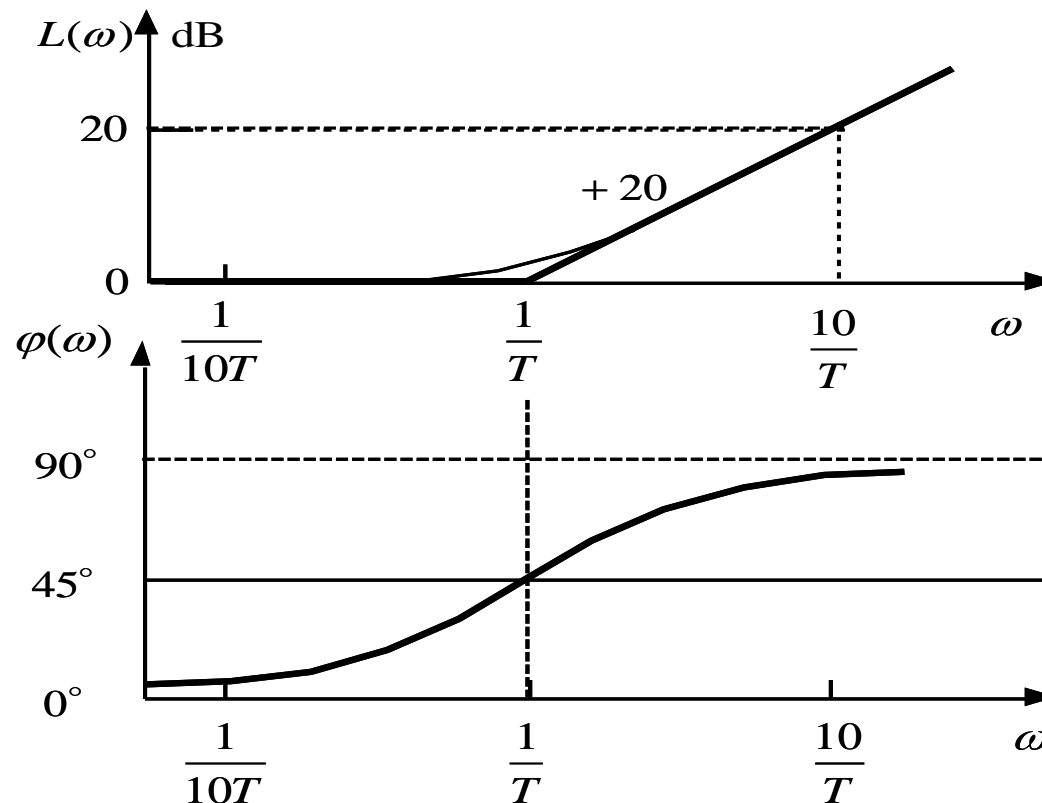
$$G(j\omega) = 1 + j\omega T \quad |G(j\omega)| = \sqrt{1 + \omega^2 T^2}$$

One order Differential element is one order the **integral** element of the reciprocal, their slope and phase displacement is just **a minus**.

$$20\lg|1 + j\omega T| = 20\lg\sqrt{1 + \omega^2 T^2} \quad \varphi(\omega) = -\tan^{-1}\omega T$$



# 5-3-1 The Typical element Bode Diagram



The slope of one order Differential element high frequency **asymptote** is +20dB/dec , the phase change from  $0^\circ$  (  $\omega=0$  ) to  $+45^\circ$  and to  $90^\circ$  (  $\omega=\infty$  )



# 5-3-1 The Typical element Bode Diagram

## 6. The Oscillation element (振荡环节)

$$G(j\omega) = \frac{1}{T^2(j\omega)^2 + 2\zeta Tj\omega + 1}$$

### The amplitude frequency characteristics

$$L(\omega) = -20\lg\sqrt{(1 - T^2\omega^2)^2 + (2\zeta T\omega)^2}$$

### The phase frequency characteristics

$$\varphi(\omega) = -\text{tg}^{-1}\left[\frac{2\zeta T\omega}{1 - T^2\omega^2}\right]$$

### Low frequency , when $\omega T \ll 1$ :

$$L(\omega) = -20\lg 1 = 0 \text{ dB}$$

——the low frequency asymptote is a  
Horizontal line with 0dB.



# 5-3-1 The Typical element Bode Diagram

High frequency , when  $\omega T \gg 1$ :

$$L(\omega) = -20 \lg \sqrt{(1 - T^2 \omega^2)^2 + (2\zeta T \omega)^2}$$

$$L(\omega) = -20 \lg(\omega^2 T^2) = -40 \lg(\omega T)$$

When  $\omega$  increase 10 time

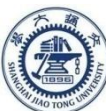
$$L(\omega) = -40 \lg 10 T \omega = -40 - 40 \lg T \omega$$

the low frequency asymptote is a Horizontal line  
with -40dB

when  $\omega = \omega_n = \frac{1}{T}$

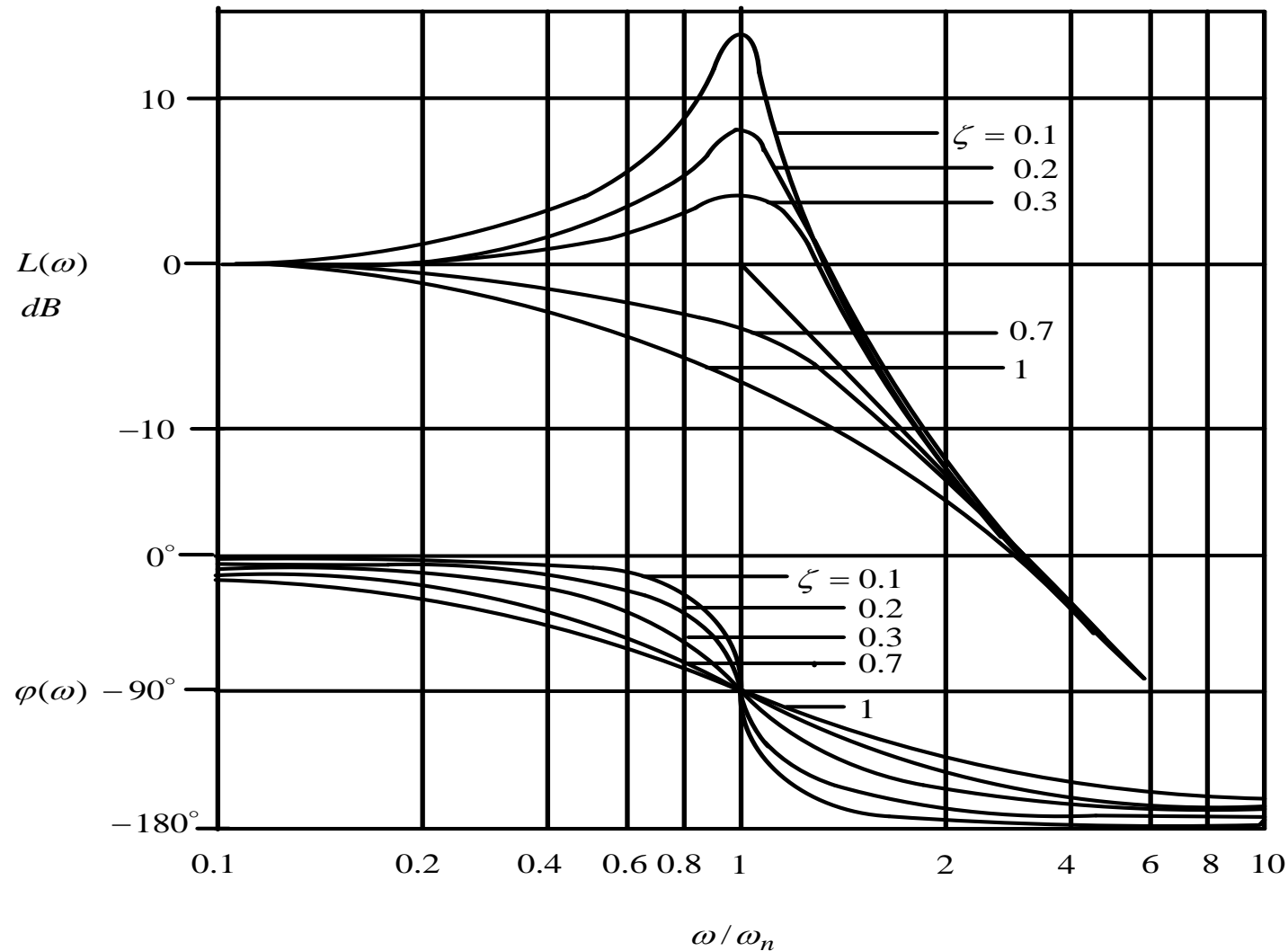
$$L(\omega) = -40 \lg \omega T = -40 \lg 1 = 0(dB)$$

That means  $\omega = \omega_n = \frac{1}{T}$  is the turn frequency of the two  
order Oscillation element .





# 5-3-1 The Typical element Bode Diagram



# 5-3-1 The Typical element Bode Diagram

- ◆ When  $\omega = \omega_n$  , will produce the resonance peak.
- ◆ The size of the damping ratio determines the resonance peak amplitude
- ◆ Phase  $\varphi$  is the function of  $\omega$  and  $\zeta$ 
  - ◆ When  $\omega=0$  ,  $\varphi(\omega)=0$
  - ◆ When  $\omega = \omega_n$  ,  $\varphi(\omega)=-180^\circ$  , have no relationship with  $\zeta$
  - ◆ When  $\omega=\infty$  ,  $\varphi(\omega)=-90^\circ$
- ◆ **phase curves is antisymmetric (斜对称)**  
correspond to curves point  $-90^\circ$



# 5-3-1 The Typical element Bode Diagram

- ◆ the resonance peak  $|G(j\omega_r)|$  on  $\omega_n = \frac{1}{T}$  can be calculated with:

The amplitude frequency characteristics:

$$|G(j\omega)| = \frac{1}{\sqrt{(1-T^2\omega^2)^2 + (2\zeta T\omega)^2}} = \frac{1}{\sqrt{g(\omega)}}$$

Where  $g(\omega) = (1-T^2\omega^2)^2 + (2\zeta T\omega)^2$

When resonance peak produced,  $|G(j\omega)| \rightarrow \text{MAX};$

$g(\omega) \rightarrow \text{MIN},$

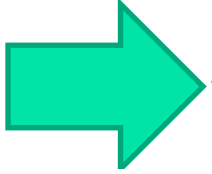
SO: 
$$\frac{dg(\omega)}{d\omega} = \frac{d}{d\omega} \left[ (1-T^2\omega^2)^2 + (2\zeta T\omega)^2 \right] = 0$$



# 5-3-1 The Typical element Bode Diagram

$$\omega_r = \frac{1}{T} \sqrt{1 - 2\zeta^2} = \omega_n \sqrt{1 - 2\zeta^2} \quad \left( 0 \leq \zeta \leq \frac{1}{\sqrt{2}} \right)$$

$$\omega_n = \frac{1}{T}$$

◆ Use  $\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$  to calculate  $\frac{d^2 g(\omega)}{d\omega^2}$    $\frac{d^2 g(\omega)}{d\omega^2} > 0$

So the resonance peak  $M_r = \left| G(j\omega_r) \right| = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}$

Resonance frequency  $\omega_r$  and resonance peak  $M_r$  all have relationship with  $\zeta$ .



# 5-3-1 The Typical element Bode Diagram

- ◆  $\zeta$  decrease,  $\omega_r \rightarrow \omega_n$ ,  $M_r$  increase
- ◆ When  $\zeta > 0.707$ ,  $\omega_r$  is a imaginary number , there is NO resonance peak  $M_r$  , the amplitude frequency characteristics drab attenuation
- ◆ When  $\zeta = 0.707$ ,  $\omega_r = 0$  ,  $M_r = 1$ 。
- ◆ When  $\zeta < 0.707$ ,  $\omega_r > 0$  ,  $M_r > 1$ 。  $\zeta \rightarrow 0$  ,  $\omega_r \rightarrow \omega_n$  ,  $M_r \rightarrow \infty$ 。
- ◆ When the resonance peak is produced , the phase of  $G(j\omega)$ :

$$G(j\omega_r) = -\text{tg}^{-1} \frac{\sqrt{1-2\zeta^2}}{\zeta} = -90^\circ + \sin^{-1} \frac{\zeta}{\sqrt{1-2\zeta^2}}$$



# 5-3-1 The Typical element Bode Diagram

## 7. Two order Differential element

Its frequency characteristics

$$G(j\omega) = 1 + 2\zeta T(j\omega) + T^2(j\omega)^2$$

The amplitude frequency characteristics

$$L(\omega) = 20 \lg \sqrt{(1 - T^2 \omega^2)^2 + (2\zeta T \omega)^2}$$

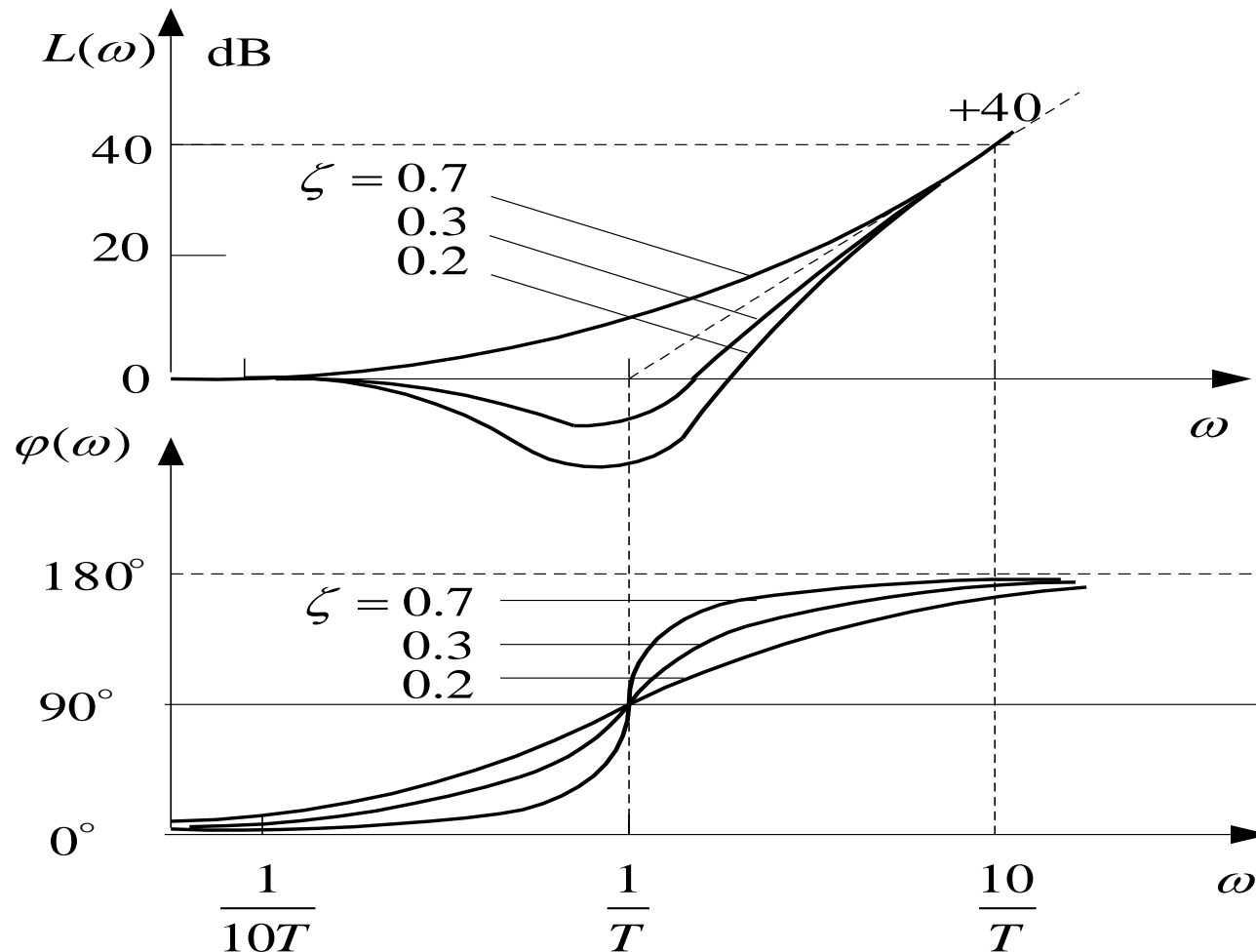
The phase frequency characteristics

$$\varphi(\omega) = \operatorname{tg}^{-1} \frac{2\zeta T \omega}{1 - T^2 \omega^2}$$

That is the second order differential element amplitude frequency and phase frequency characteristics respectively with the oscillation of the corresponding link is about the **horizontal axis symmetry characteristics**. At this time, the logarithm amplitude frequency characteristics of the slope of the high frequency between for + 40 dB/dec and phase frequency from 0 ° (corresponding  $\omega = 0$ ) by 90 °, last tends to 180 ° ( $\omega \rightarrow \infty$ )



# 5-3-1 The Typical element Bode Diagram



# 5-3-1 The Typical element Bode Diagram

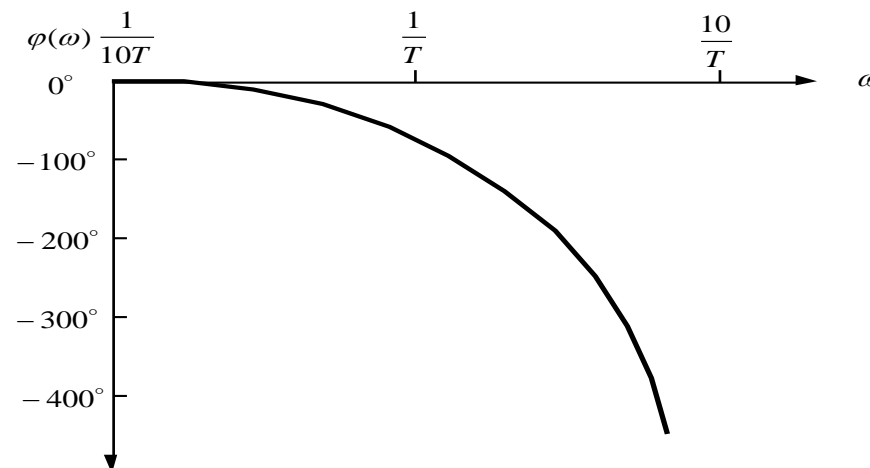
## 8. The Delay element

Its frequency characteristics  $e^{-j\omega T}$

The amplitude and phase frequency characteristics are :

$$L(\omega) = 20\lg|G(j\omega)| = 20\lg 1 = 0 \text{ dB} \quad \phi(\omega) = \angle e^{-j\omega T} = -\omega T(\text{rad}) = -57.3^\circ \omega T$$

the phase and frequency  $\omega$  have a linear relationship





## 5-3-2 How to plot the Bode Diagram

### Basic steps :

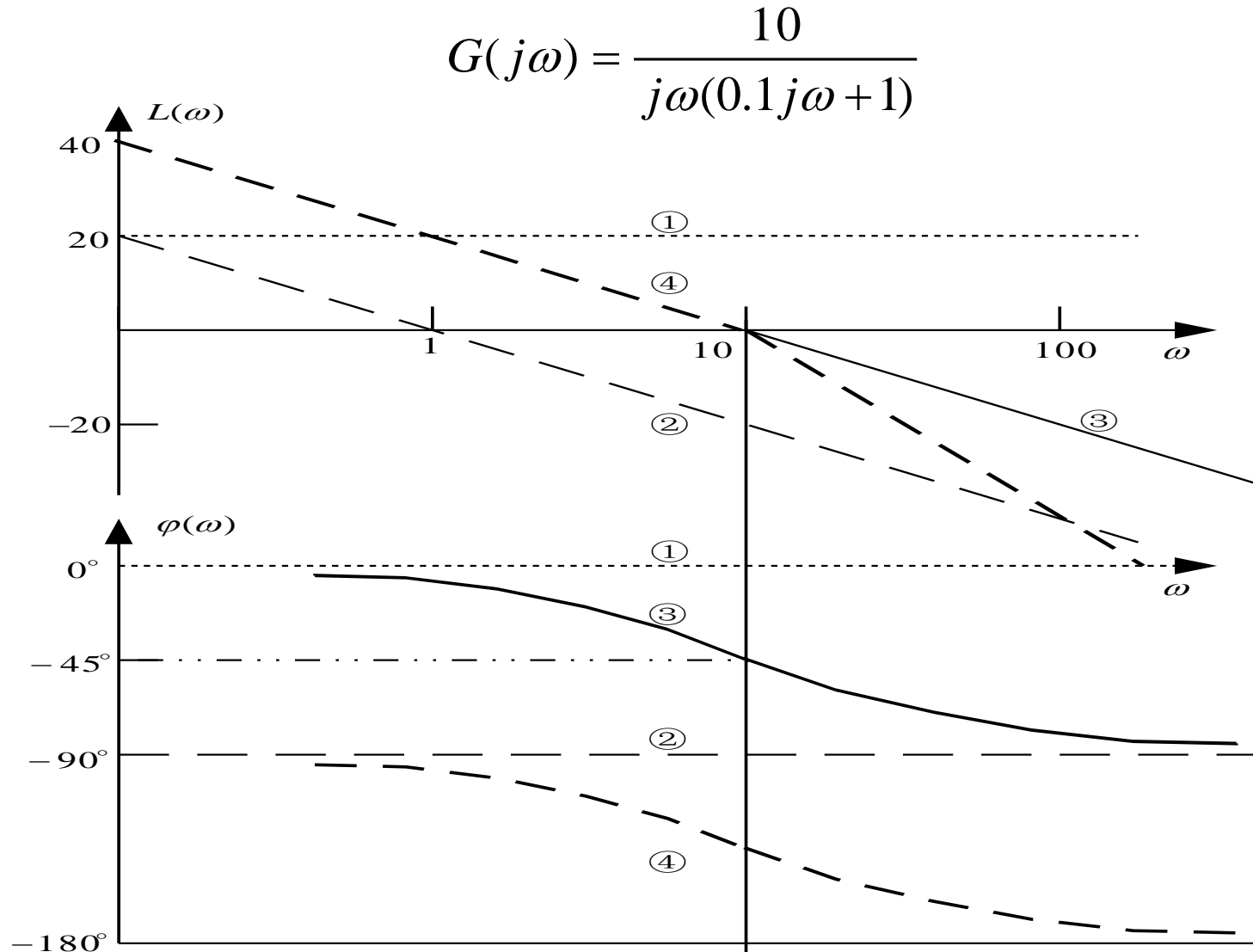
The system frequency characteristics of various typical element rewritten to the product of form, draw every element of the logarithm amplitude frequency and phase frequency curve, then stack with frequency, that is, get this system Bode figure.

Example1 :

$$G(j\omega) = \frac{10}{j\omega(0.1j\omega + 1)}$$



# 5-3-2 How to plot the Bode Diagram



## 5-3-2 How to plot the Bode Diagram

### ◆ Summary of Bode plot Rules(1)

1. Manipulate the transfer function into the Bode form as follows:

$$KG(j\omega) = K_0 \frac{(j\omega_{T1} + 1)(j\omega_{T2} + 1) \cdots}{(j\omega_{Ta} + 1)(j\omega_{Tb} + 1) \cdots}$$

2. Determine the value of  $n$  for the  $K_0(j\omega)^n$  term. Plot the low frequency magnitude asymptote through the point  $K_0$  at  $\omega = 1$  with a slope of  $n$  (or  $n \times 20$  db per decade)
3. Complete the composite magnitude asymptote .



# 5-3-2 How to plot the Bode Diagram

## ◆ Summary of Bode plot Rules(2)

4. Sketch in the approximate magnitude curve;
5. Plot the Low-frequency asymptote of the phase curve  $\phi = n \times 90^\circ$
6. Sketch in the approximate phase curve by changing the phase  $\pm 90^\circ$  or  $\pm 180^\circ$  at each break point in ascending order.
7. Locate the asymptote for each individual phase curve so that their phase change corresponds to steps in the phase toward or away from the approximate curve indicated by step 6.



# 5-3-2 How to plot the Bode Diagram

- ◆ Summary of Bode plot Rules(3)
  - 8. Graphically add each phase curve  
(all steps are describe in detail at Page 250)

Analysis the Example .

$$G(s) = \frac{3S + 12}{S(50S^2 + 10S + 2)}$$



# 5-3-2 How to plot the Bode Diagram

- ◆ Advantages of working with Frequency Response in terms of Bode Plots
  - ◆ Dynamic compensator design can be based entirely on Bode plots.
  - ◆ Bode plots can be determined experimentally.
  - ◆ Bode plots of systems in series (or tandem) simply add, which is quite convenient.
  - ◆ The use of a log scale permits a much wider range of frequencies to be displayed on a single plot than is possible with linear scales.



## ◆ HOMEWORK

### ◆ Page 310

- Ex6.1

- ◆ Ex6.2

- ◆ Ex6.3

### ◆ Page 311-312

- Ex6.8

- Ex6.9

- ◆ Deadline:NOV.27.2012



# Principles of Automatic Control

## - Chap 5 The Frequency-Response Design Method (5-4 Stability Analysis)

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# Schedule for following time

- 12<sup>th</sup> week (Nov.26,28) Chap5.5/5.6
- 13<sup>th</sup> week (Dec.3,5) Course Design Chap5
- 14<sup>th</sup> week (Dec.10,12) Chap6
- 15<sup>th</sup> week (Dec.17,19) Course Design Chap6
- 16<sup>th</sup> week (Dec.24) General Review for ACT
- 16<sup>th</sup> week (Dec.27) Final Exam



# Chap 5 The Frequency-Response Design Method

- ◆5-1 Frequency Characteristics Definition
- ◆5-2 Nyquist Diagram
- ◆5-3 Bode Diagram
- ◆5-4 **Stability Analysis**
- ◆5-5 Frequency domain of steady-state analysis
- ◆5-6 Frequency domain dynamic analysis



# Chap 5 The Frequency-Response Design Method

Review:

## Linear time-invariant system stability criterion

### ◆ Time domain: **The Routh-Hurwitz Stability**

**Criterion** (Chap3.4: a system is stable if all the poles of the transfer function are in the left-hand plane; to built **the Routh-Hurwitz criterion table** )

### ◆ Complex domain: System **stability** Analysis Using the **Root**

**Locus** (Chap4.3 the root is distributed in S left plane, system is stable , and vice is unstable)

### ◆ Frequency domain: System **stability** Analysis Using the **Open loop Frequency Response** (chap5.3:The Nyquist Stability Criterion)



# Chap 5 The Frequency-Response Design Method

- ◆ Review :The relationship of zero and pole between the open-loop and close-loop system

Set the **open-loop system**:

$$G(s)H(s) = P(s) / Q(s)$$

So the characteristics function of **close-loop system**:

$$F(s) = 1 + G(s)H(s) = \frac{P(s) + Q(s)}{Q(s)} = \frac{K_r \prod_{i=1}^n (s + z_i)}{\prod_{j=1}^n (s + z_j)}$$

The close-loop transfer function:

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{F(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{G(s) \bullet Q(s)}{P(s) + Q(s)}$$



# Chap 5 The Frequency-Response Design Method

- ◆ Conclusion for the relationship of **system stability** with the **open-loop transfer function** and **close-loop transfer function**
  1. **The open-loop stability** depends on the **poles** of  $G(s)H(s)$  , that is the root of the polynomial  $Q(s)$
  2. **The close-loop stability** depends on the **poles** of  $M(s)$  , or the zero of characteristics function of **close-loop system**  $F(s)$  , that is the root of the polynomial  $P(s) + Q(s)$
  3. There are NO necessarily link between the the **open-loop stability** and **The close-loop stability**.



# Chap 5 The Frequency-Response Design Method

- ◆ The Nyquist Stability Criterion is based on a result from complex variable theory known as **the argument principle** (辐角原理).
- ◆ The Nyquist stability Criterion relates the open-loop frequency response to the number of closed-loop poles of the system in the **RHP(Right half-plane)**.
- ◆ Study of the Nyquist criterion will allow us to determine stability from the frequency response of a complex system, perhaps with one or more resonances, where the **magnitude curves crosses 1 several times** and/or **the phase crosses  $180^\circ$  several times**.
- ◆ The Nyquist Stability Criterion is very useful in dealing with open-loop, unstable systems, nonminimum-phase system/minimum-phase system, and system with pure delays.



# 5-4-1 the argument principle

By complex-variable function, it is known that the plane to  $S$  except the singularity point, after complex-variable function  $F(S)$  mapping, in  $F(S)$  plane can be found on the corresponding function.

Set auxiliary function:

$$F(s) = \frac{\prod_{i=1}^n (s + z_i)}{\prod_{j=1}^n (s + p_j)} = \frac{\prod_{i=1}^n |s + z_i|}{\prod_{j=1}^n |s + p_j|} \left( \sum_{i=1}^n \angle s + z_i - \sum_{j=1}^n \angle s + p_j \right)$$

$$= |F(s)| \angle F(s)$$



# 5-4-1 the argument principle

**Set** : S from  $s_1$  began to any closed path along the  $\Gamma_s$  (not pass  $F(s)$  of zero and poles) **clockwise a circle**,  $F(s)$  phase angle of the changes are as follows:

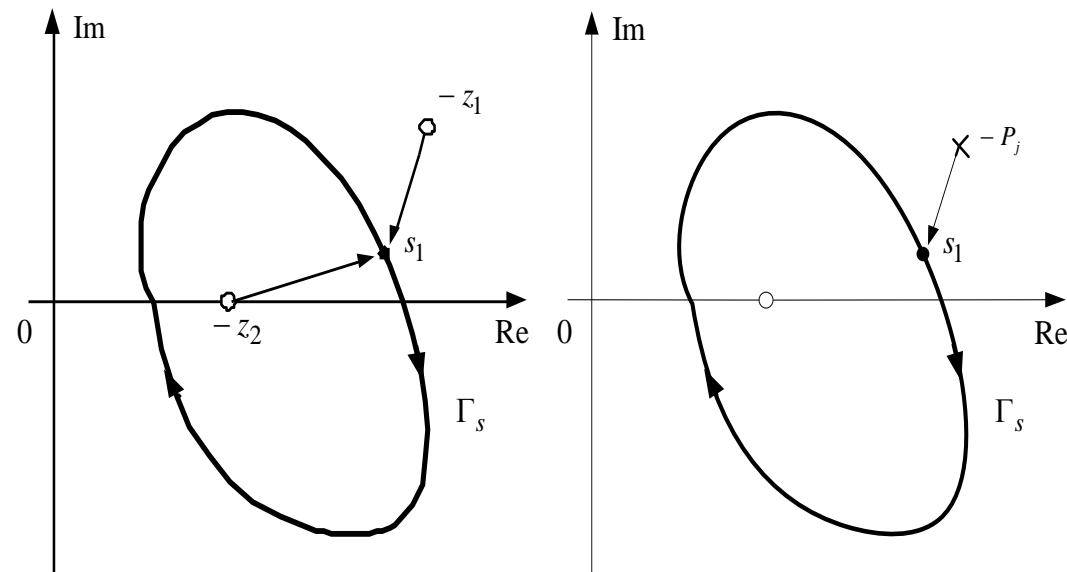
**Zero ( $-Z_i$ )      Poles( $-P_j$ )**

1)  $-Z_i$  outside of  $\Gamma_s$

$$\Delta \angle s + z_i = 0^\circ$$

2)  $-P_j$  outside of  $\Gamma_s$

$$\Delta \angle s + p_j = 0^\circ$$



**conclusion** : phase angle is not change





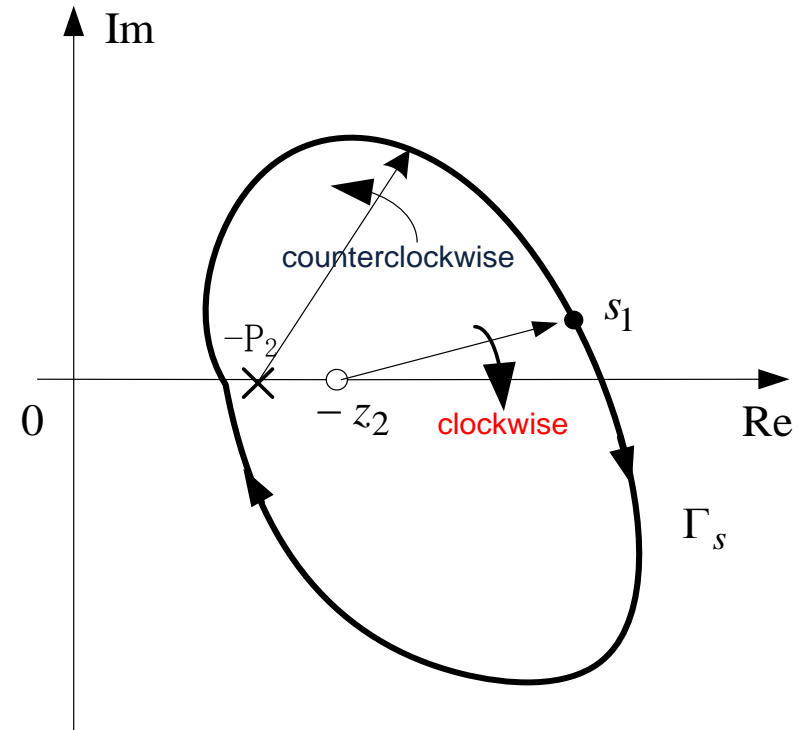
# 5-4-1 the argument principle

1)  $-Z_i$  inside of  $\Gamma_s$  (**clockwise**)

$$\Delta \angle s + z_i = -2\pi$$

2)  $-P_j$  inside of  $\Gamma_s$   
(**counterclockwise**)

$$\Delta \angle s + p_j = 2\pi$$



**Conclusion:** if  $F(s)$  in  $\Gamma_s$ , there are  $Z$  zero and  $P$  pole, then, when the  $S$  along the  $\Gamma_s$  clockwise one lap,  $F(s)$  **phase angles change (clockwise)** :

$$\angle F(s) = -2\pi(Z - P)$$



# 5-4-1 the argument principle

## the argument principle :

$F(s)$  is a rational function of a single value  $s$ , in a plane closed path surrounded the  $F(s)$ , there are  $Z$  zero and  $P$  pole, and  $F(s)$  not through any one of the zero and poles, then, when the  $s$  closed path along the **clockwise one lap**, mapping to  $F(s)$  plane within the  $F(s)$  curve clockwise around the origin ( $Z-P$ ) circle. That is :

$$N = Z - P$$

(or counterclockwise around origin point is :  $N = P - Z$ )

among them:  $N$  for laps, positive and negative for the direction of rotation: **counterclockwise is positive; clockwise is negative\***.

\*Note: depend on different reference books.

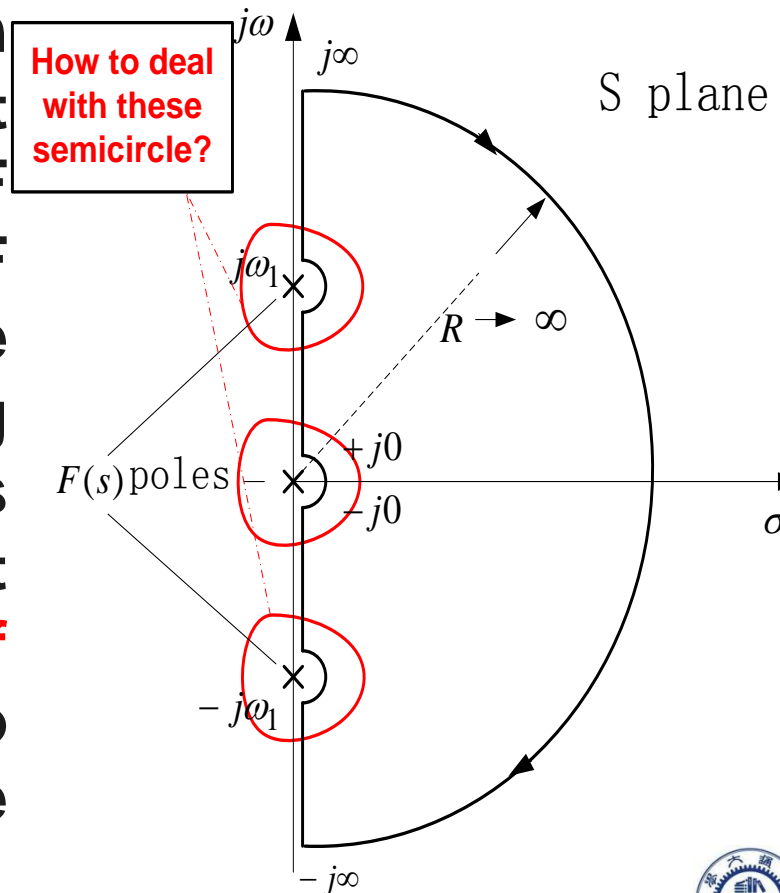


# 5-4-2 The Nyquist Stability Criterion

## 1 . Nyquist path

$$s = -j\infty \rightarrow -j0 \rightarrow +j0 \rightarrow +j\infty \rightarrow -j\infty$$

**Clockwise** direction surrounded the whole  $s$  right plane. Because not through the  $F(s)$  of any zero, poles, so when  $F(s)$  with several poles in  $s$  plane the imaginary axis (including origin point), set these points as the central point of circle, and set the **infinitesimal radius of semicircle**, according to **counterclockwise** around these points from the right.



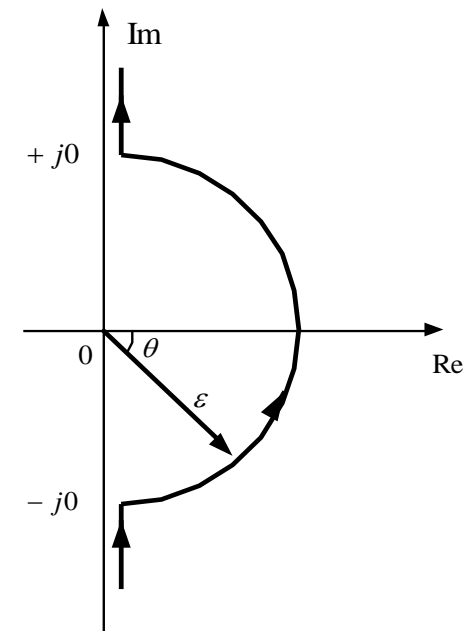
# 5-4-2 The Nyquist Stability Criterion

a. when  $s=0$  is the poles of open loop .

Nyquist path:

$s = -j0 \rightarrow +j0$ . set the origin point as **the central point of**, the infinitesimal radius  $\varepsilon$  of semicircle, according to counterclockwise around the origin from the right.

Set  $s = \varepsilon e^{j\theta}$ ,  $\varepsilon \rightarrow 0$ , when  $s = -j0$  tend to  $+j0$ ,  $\theta$ :  $-90^\circ$  to  $+90^\circ$  ( I type system )



$$G(s)H(s) \Big|_{s=\varepsilon e^{j\theta}} = \frac{K \prod_{i=1}^m (\tau_i \varepsilon e^{j\theta} + 1)}{(\varepsilon e^{j\theta})^\nu \prod_{j=\nu+1}^n (T_j \varepsilon e^{j\theta} + 1)} = \frac{K}{(\varepsilon e^{j\theta})^\nu} = \infty e^{-j\nu\theta}$$

so  $\angle G(j\omega)H(j\omega) : \nu \times (-90^\circ)$  to  $\nu \times 90^\circ$  (there are  $\nu$  poles).



# 5-4-2 The Nyquist Stability Criterion

**so:** when  $s: -j0$  tend to  $+j0$ , the Nyquist curve of  $G(s)H(s)$  is radius of  $\infty$ , Clockwise around  $\nu\pi$ ,

**b .** the nyquist curve when  $s \rightarrow \infty$

set:  $R \rightarrow \infty$ , so

$$G(s)H(s) \Big|_{s=Re^{j\theta}} = \frac{K_1 \prod_{i=1}^m (Re^{j\theta} + z_i)}{\prod_{j=1}^n (Re^{j\theta} + p_j)} = \varepsilon e^{-j(n-m)\theta}$$

so, when  $n - m > 0$ ,  $\varepsilon$  tend to 0.

$\angle G(j\omega)H(j\omega)$  :  $-(n-m) 90^\circ$  tend to  $+(n-m) 90^\circ$



# 5-4-2 The Nyquist Stability Criterion

## 2. The Nyquist Stability Criterion

set :  $F(s) = 1 + G(s)H(s)$  — the characteristics function of **close-loop system**

**so: the Zero of  $F(s)$  is the poles of close-loop system**

### **(1) The system stability analysis on $1 + G(S)H(S)$ plane**

If  $S$  along the Nyquist path loops around one time, according to the argument principle,  $F(s)$  on the plane drawings of  $F(s)$  curve  $\Gamma_F$  **clockwise** around the lap of the origin is  $N$ , which is equal the  $F(s)$  in  $s$  right open plane within the pole number of  $P$  minus the zero number of  $Z$ :

$$N = Z - P(\text{clockwise}) \quad \text{Note: } N = P - Z (\text{counterclockwise})$$

**When  $Z = 0$ , the closed-loop system transfer function that no pole in  $S$  right open plane, the system is stable, Conversely, the system is not stable.**



# 5-4-2 The Nyquist Stability Criterion

( 2 ) The system stability analysis on  $G(s)H(s)$  plane - **The Nyquist Stability Criterion**

Because of there is difference between  $1+ G(s)H(s)$  and  $G(s)H(s)$  is 1 , so Stability Criterion can be defined as :

**The Nyquist Stability Criterion** : The **Sufficient and Necessary Criterion** for Stabilization of the closed-loop system : s along the Nyquist path loops around one time,  $G(j\omega)H(j\omega)$  curve counterclockwise around  $(-1, j0)$  point P circle.

Here, P for  $G(s)H(s)$  is located in the plane of number right s pole.



# 5-4-2 The Nyquist Stability Criterion

- A. if  $P = 0$ , and  $N = 0$ , that is GH curve is not surrounded  $(-1, j0)$  point, the closed-loop system is stable;
- B. if  $P \neq 0$ , and  $N = P$ , that is GH curve **clockwise** around  $(-1, j0)$  point  $P$  circle, the closed-loop system is stable, otherwise the system is not stable. Unstable system distribution in the number of poles right s plane can calculate as:  
$$Z = N - P$$
- C. If GH curve through the  $(-1, j0)$  point  $L$  times, show closed loop system has a  $L$  poles distribution in the imaginary axis in plane  $s$ .





# 5-4-2 The Nyquist Stability Criterion

**EXAMPLE** : Try judgment system stability

$$G(j\omega)H(j\omega) = \frac{K_1(j\omega - 1)}{j\omega(j\omega + 1)}$$

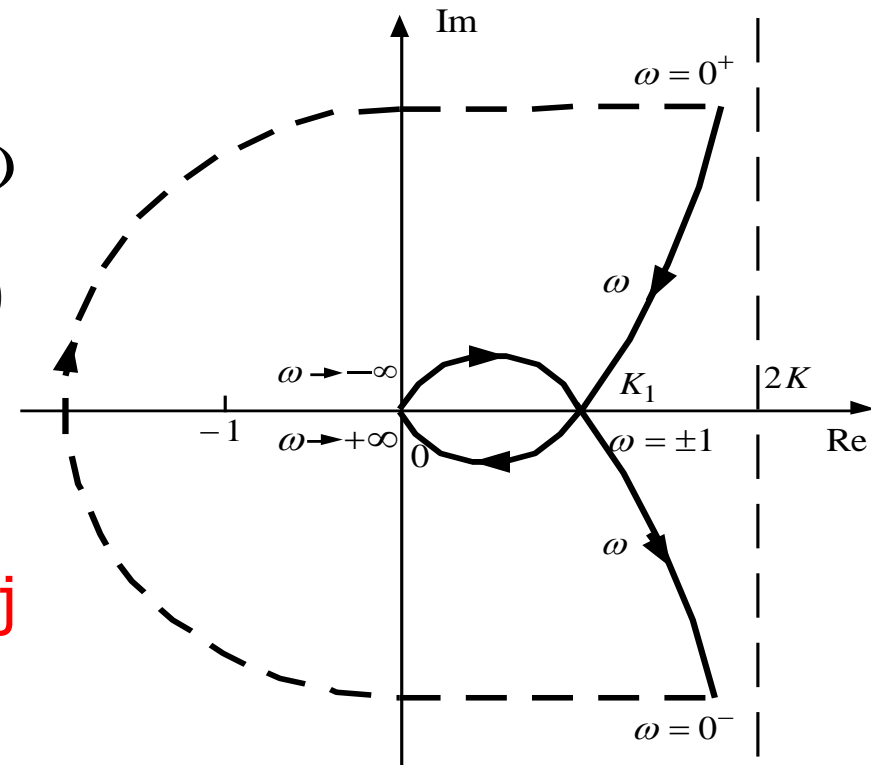
Solution:

$$G(s)H(s) = \frac{K_1(s - 1)}{s(s + 1)} \quad (K_1 > 0)$$

Firstly, draw the  $G(j\omega)H(j\omega)$

Nyquist curve as  $+j0$  to  $+j\infty$ .

According to the symmetry properties, to draw the  $G(j\omega)H(j\omega)$  Nyquist curve as  $-j0$  to  $-j\infty$ .

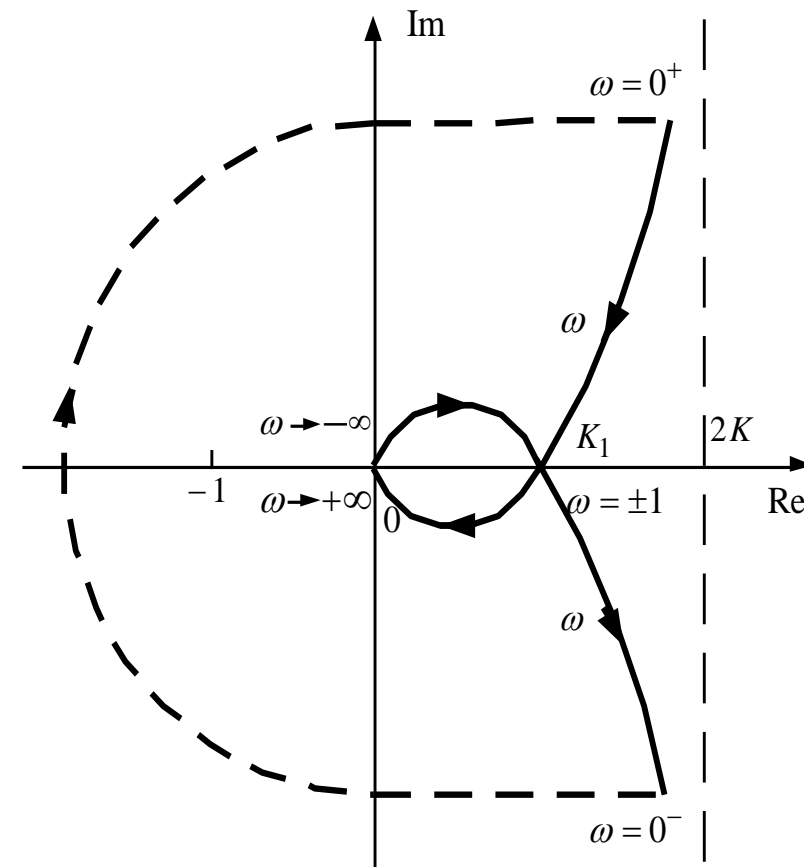


# 5-4-2 The Nyquist Stability Criterion

In this example,  $\nu = 1$ , that is when the  $S$  from  $-j0$  turned to  $+j0$ ,  $G(j\omega)H(j\omega)$  curve has a infinite radius, clockwise around  $\pi$  Angle (dotted line). And can be obtained, when  $\omega = \pm 1$ , the cross point with the real axis is  $K_1$  (Seen from the chart).  $G(s)H(s)$  Nyquist curve clockwise around  $(-1, j0)$  take a turn. Here,  $N = 1$ , and because  $P = 0$ , so:

$$Z = P - N = 1;$$

It means that is **an unstable system**, there is a closed loop poles in  $s$  right plane



# 5-4-2 The Nyquist Stability Criterion

◆ **EXAMPLE** : Try judgment system stability

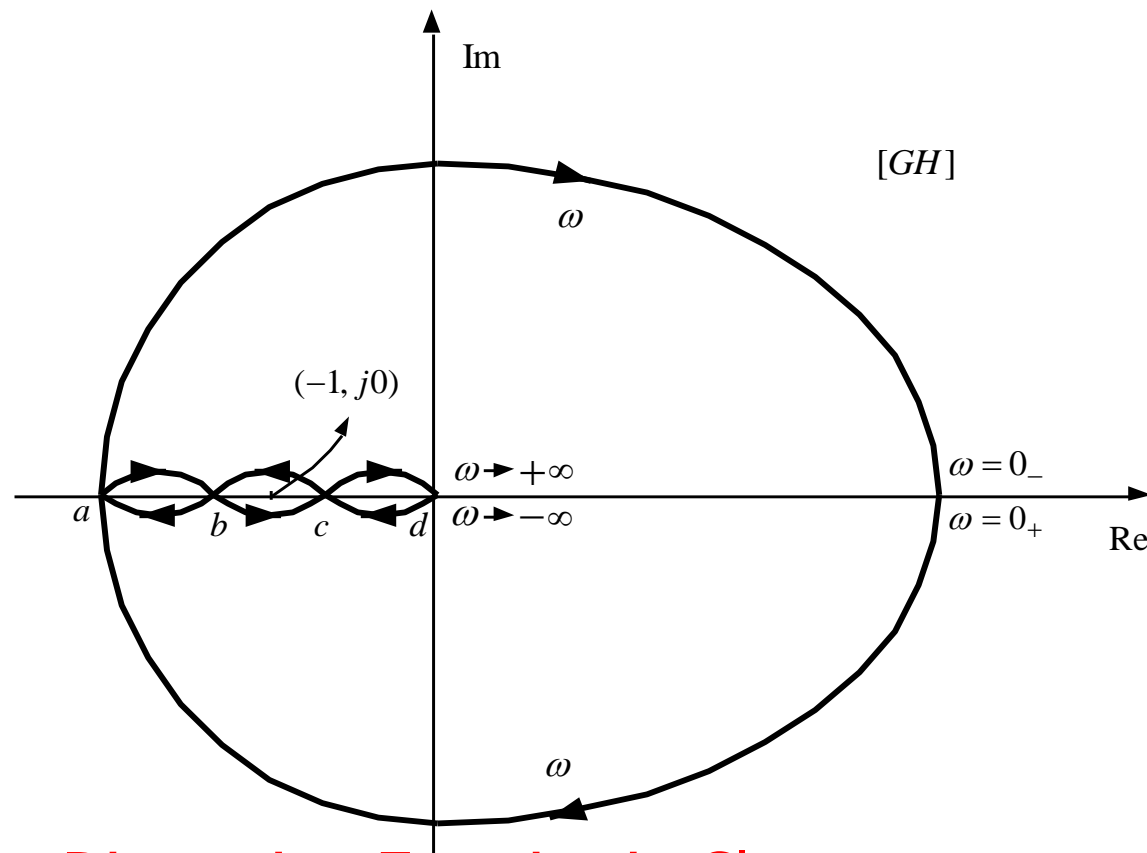
$$G(S)H(S) = \frac{K}{S(1 + T_1S)(1 + T_2S)}$$



# 5-4-2 The Nyquist Stability Criterion

**EXAMPLE :** Try judgment system stability. Set in the open loop transfer function:  $T_5 < T_1 < T_2$ ,  $T_3$  &  $T_4$

$$G(S)H(S) = \frac{K(T_1 S + 1)}{(1 + T_2 S)(1 + T_3 S)(1 + T_4 S)(S + T_5 S)}$$



Discussion Exercise in Classroom

# 5-4-2 The Nyquist Stability Criterion

- ◆ Solution: if a  $K$  value curve under  $GH$  as shown in figure, for  $N = 0$ , and  $P = 0$ , then the system stability.
- 1.  $K$  increases, make  $(-1, j0)$  is located in c, d, the curve clockwise surrounded  $(-1, j0)$  two laps, the system is not stable.
- 2.  $K$  decreases, and make  $(-1, j0)$  is located in a, b, the curve clockwise surrounded  $(-1, j0)$  point two laps, the system is still unstable. **K to reduce**, make  $(-1, j0)$  point is located in a point on the left, then the closed-loop system is stable again. This system is called conditions stable system. That is, to make the system stability,  $K$  must meet certain conditions.



# 5-4-2 The Nyquist Stability Criterion

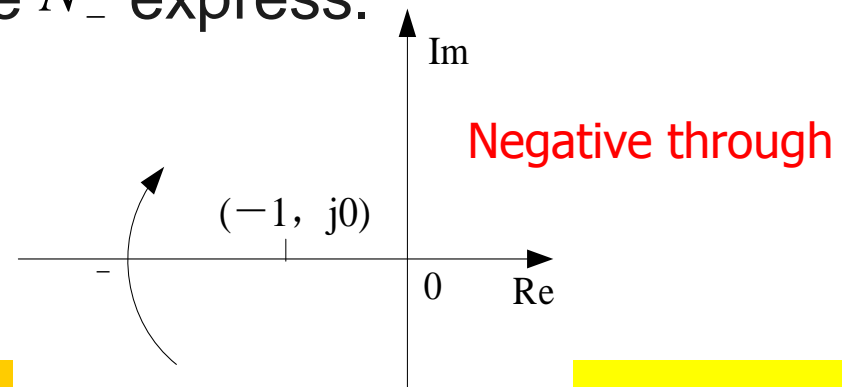
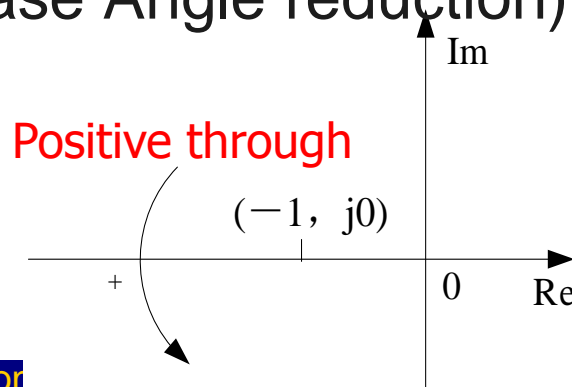
## 3 A simplified Nyquist Stability Criterion

### (1) the concept of positive and negative through

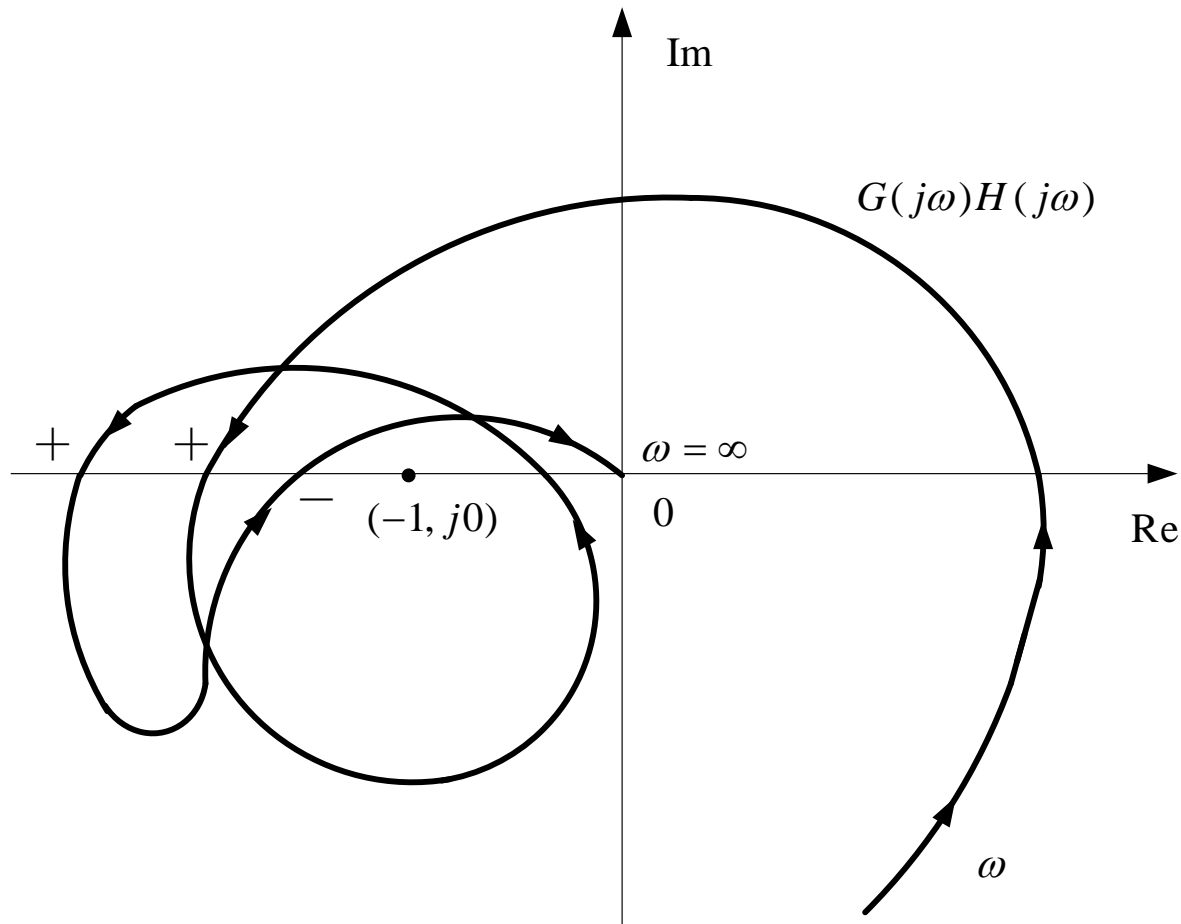
Because of  $G(j\omega)H(j\omega)$  curve is symmetrical of the real axis. In real application, we always draw the part curve only when  $\omega = 0 \rightarrow \infty$ . So, the so-called "through" refers to track through the section  $(-1, -\infty)$ .

**Positive through**: from **up to down** through one time (phase Angle increase), use  $N_+$  to express.

**Negative through**: from **bottom to up** through one time (phase Angle reduction), use  $N_-$  express.



# 5-4-2 The Nyquist Stability Criterion

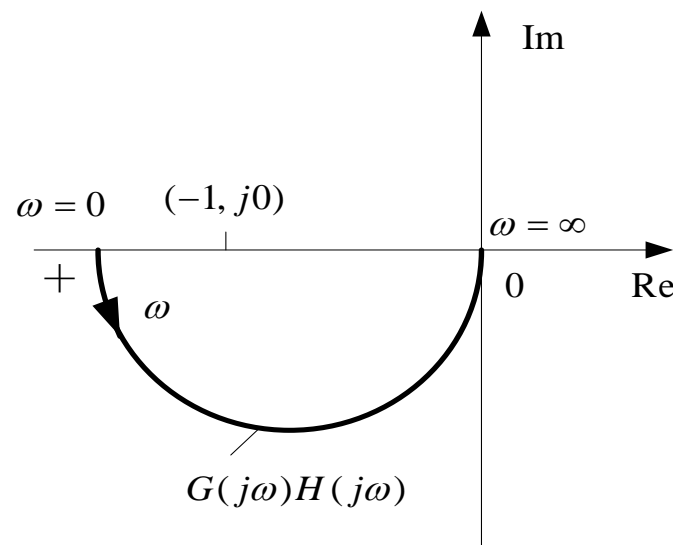


$$N_+ = 2 \quad N_- = 1$$

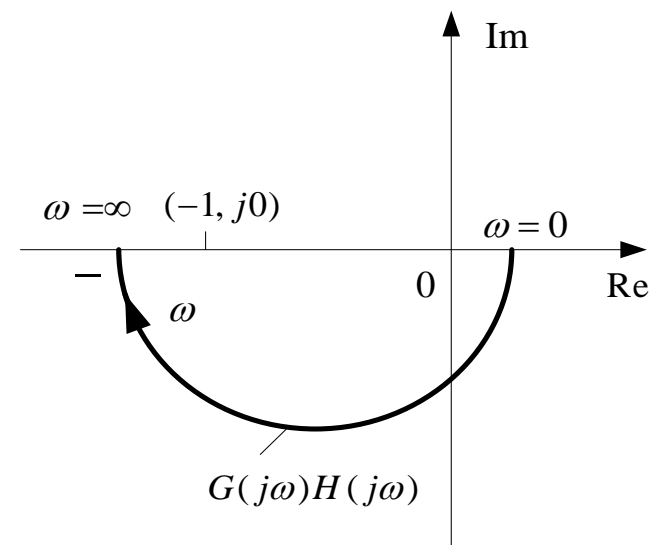


# 5-4-2 The Nyquist Stability Criterion

If the  $G(j\omega)H(j\omega)$  curves is end or begin in the left of negative axis  $(-1, j0)$ , So the times are called as the half times through, and also have **+ 1/2 times through** and **-1/2 times through**.



**+ 1/2 times through**



**-1/2 times through**





# 5-4-2 The Nyquist Stability Criterion

- ◆ If  $G(j\omega)H(j\omega)$  curves run counterclockwise surrounded the point  $(-1, j0)$  one circle, it will be **Positive through** the one time. Conversely, if a clockwise direction surrounded point  $(-1, j0)$  one circle, it will be **through a negative**. This is the sum of positive and negative through to  $G(j\omega)H(j\omega)$  surrounded lap. So **Nyquist Stability Criterion** can be expressed as follows:  
when  $\omega$  from 0 to change  $\infty$ , **the sufficient and necessary criterion** of the closed-loop system stabilization is  $G(j\omega)H(j\omega)$  curve has **the through sum** in the left of  $(-1, j0)$  points on the real axis of positive and negative is  **$P / 2$  circle**.

If the open-loop transfer function without pole distribution in  $S$  right plane,  $P = 0$ , the closed-loop system is stable, the sufficient and necessary criterion should be  $N = 0$ :

Note: corresponding range change is  $0 \rightarrow +\infty$ .

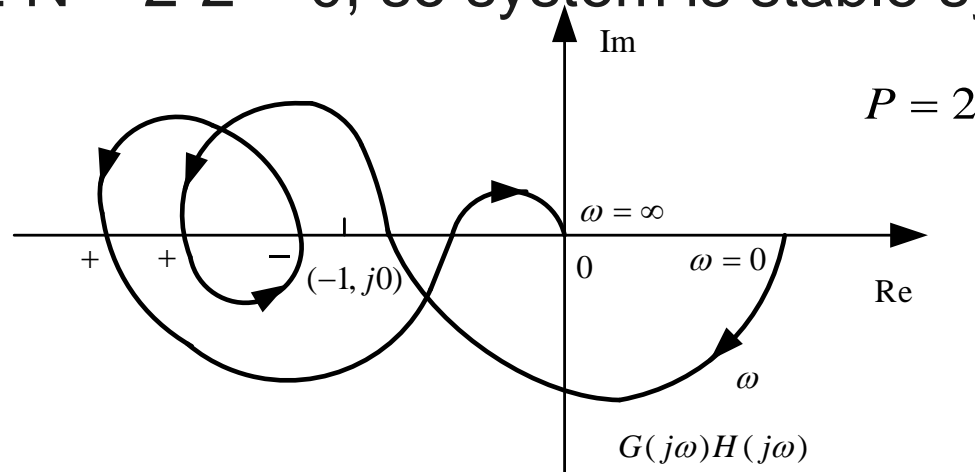


# 5-4-2 The Nyquist Stability Criterion

Example: a system  $G(j\omega)H(j\omega)$  curves as follows, are known to have two open loop poles distribution in s right plane, try judge system stability.

Solution: the system has two open loop poles distribution in s right plane ( $P = 2$ ),  $G(j\omega)H(j\omega)$  curves on a point  $(-1, j0)$  to the left of negative real axis with two positive through, one negative through, because:  $N_+ - N_- = 2 - 1 = 1$ ,

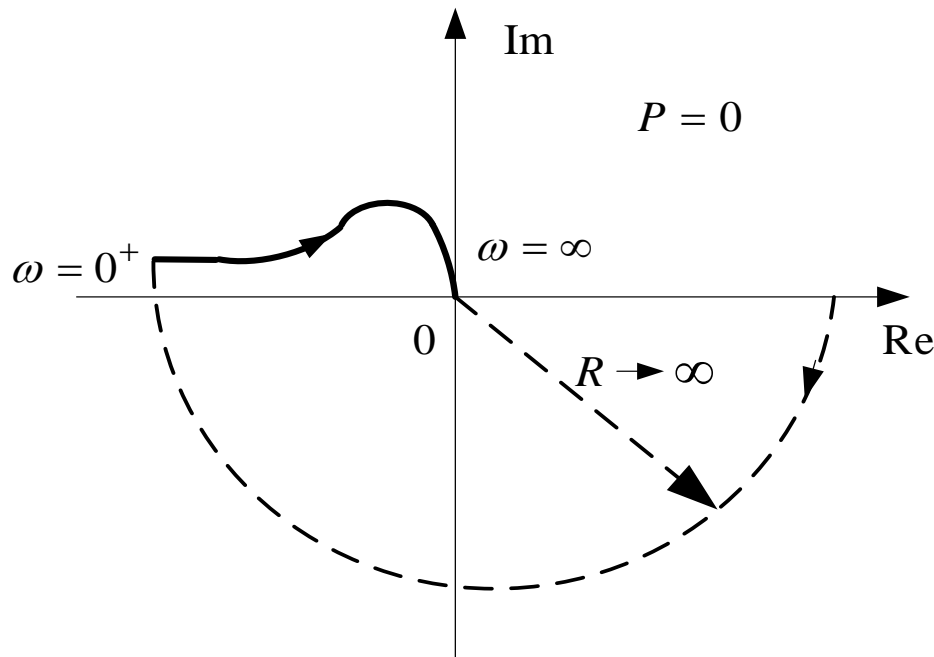
So:  $Z = P - 2N = 2 - 2 = 0$ , so system is stable system



# 5-4-2 The Nyquist Stability Criterion

**Ex:** Try judgment system stability .

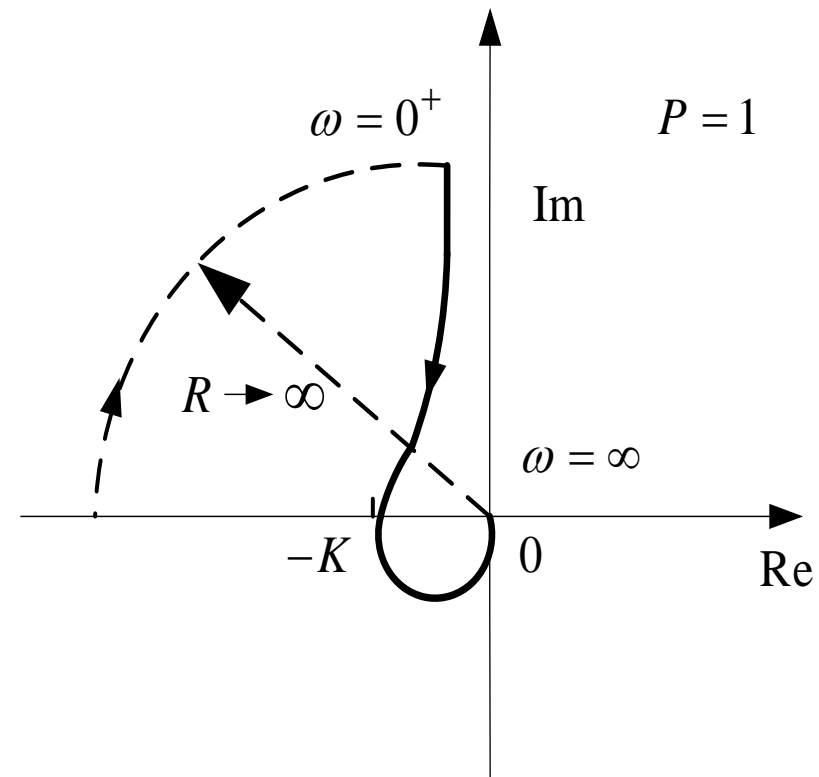
**S:** (a) :  $N = N_+ - N_- = (0 - 1) = -1$  , and  $P = 0$  , so  $Z = P - 2N = 2$  , system is instability.



# 5-4-2 The Nyquist Stability Criterion

(b) :  $K > 1$  ,  $N = N_+ - N_- = 1 - 1/2 = -1/2$  ,  
when  $P=1$  , So  $Z = P - 2N = 0$ , the  
closed-loop system is stable;

$K < 1$  ,  $N = N_+ - N_- = 0 - 1/2 = -1/2$  ,  $1/2$  ,  
when  $P=1$  , So  $Z = P - 2N = 2$  , the  
closed-loop system is unstable ;  
 $K=1$  , Nyquist curve through  $(-1, j0)$   
point two times, that there are two  
root in the imaginary axis, so the  
system is not stable.



## 5-4-3 The Nyquist Stability Criterion in Bode Diagram

### The Nyquist Stability Criterion in Bode Diagram

#### Polar Diagram

Unit circle

Unit circle within round area

Unit circle outside round area

Negative real axis

#### Bode Diagram

0db line(**magnitude curves**)

0 db line the following areas

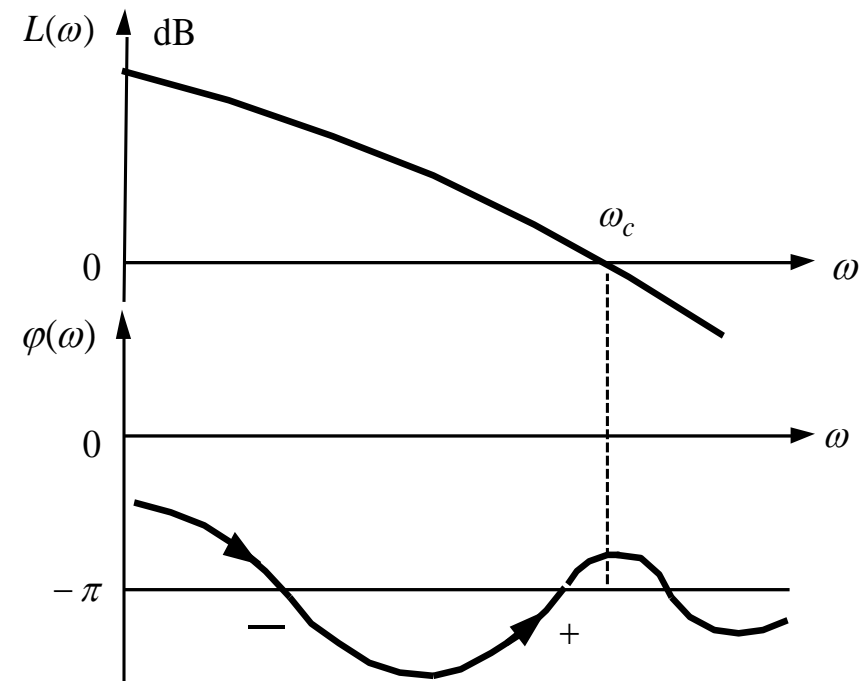
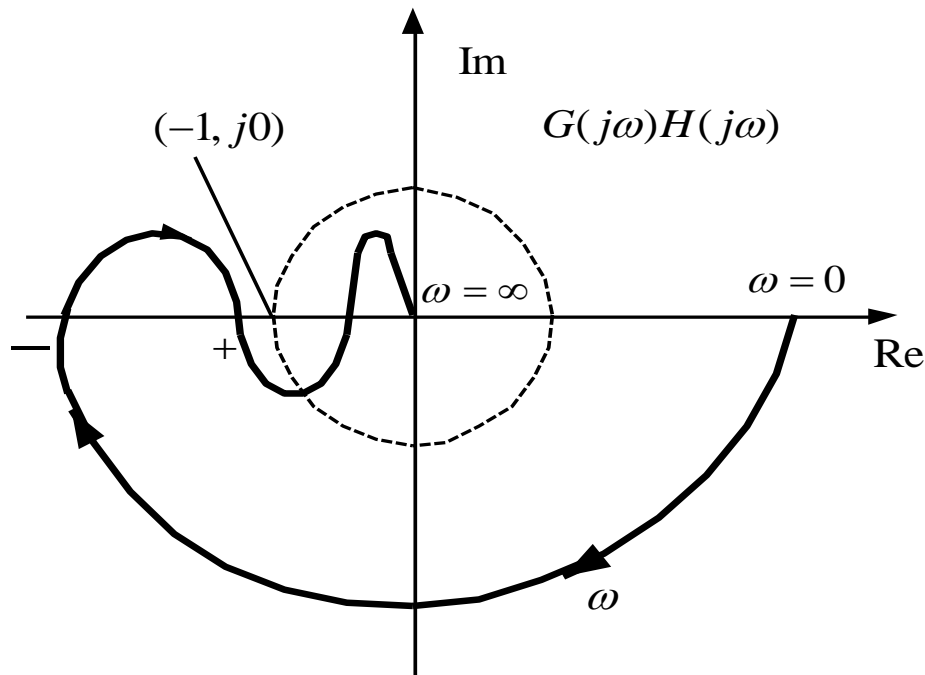
0 db line above area

-180<sup>0</sup> line(**phase curves**)

Therefore, Nyquist curve **top-down** (or bottom-up) through  $(-1, j0)$  on the left side of the point negative real axis, **equivalent to** in the **Bode Diagram** when  $L(\omega) > 0$  db phase frequency characteristics curve when **bottom-up** (or top-down) through  $-180^\circ$  line.



# 5-4-3 The Nyquist Stability Criterion in Bode Diagram



## 5-4-3 The Nyquist Stability Criterion in Bode Diagram

- ◆ With reference to the polar coordinates of Nyquist criterion definition, the Nyquist criterion in logarithmic coordinates is defined as:

**The sufficient and necessary criterion of Stabilization for the closed-loop system is :**

when  $\omega$  change from 0 to  $\infty$  , in the open loop logarithm amplitude frequency characteristics in

$L(\omega) \geq 0$  , phase frequency characteristics  $\varphi(\omega)$  through times should be  $P/2$  ( the sum of positive through and the negative through ).

here,  $P$  is the number of poles for open-loop transfer function in the  $s$  the right plane.



## 5-4-3 The Nyquist Stability Criterion in Bode Diagram

- ◆ If the open-loop transfer function without pole distribution in  $S$  right plane, that is  $P = 0$

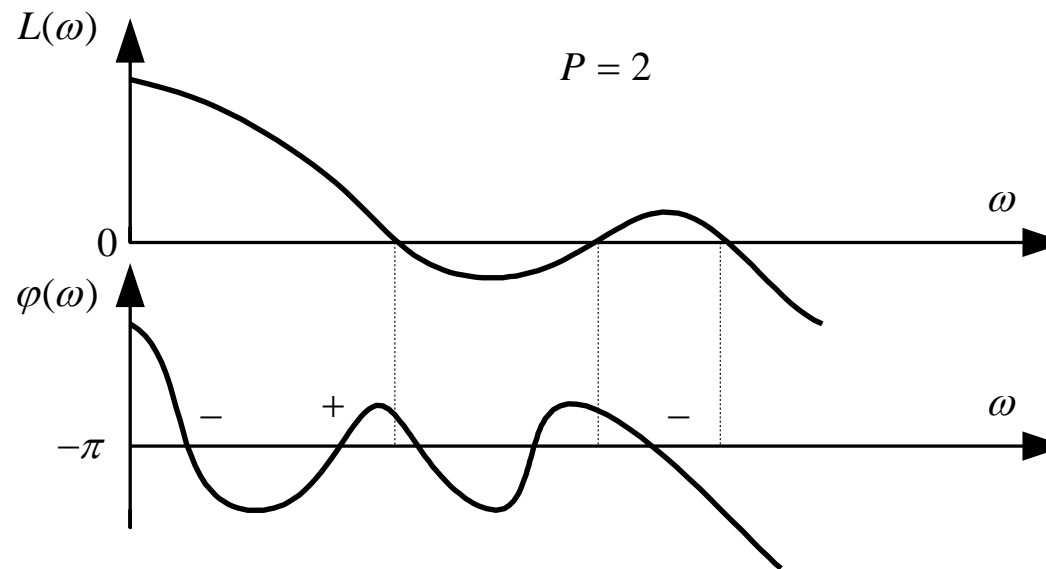
So, the sufficient and necessary criterion of Stabilization for the closed-loop system is : the magnitude curves in  $L(\omega) \geq 0$  , the algebra sum of positive and negative through for phase curves through the axis is zero. Or don't cross  $-\pi$  axis.





## 5-4-3 The Nyquist Stability Criterion in Bode Diagram

Ex : A system has two open loop poles in S right plane  
(  $P=2$  )



$N_+ - N_- = 1 - 2 = -1$  is not equal  $P/2$  (  $=1$  ), So, the system is not stable



# Homeworks

**Page:313-314**

**Ex6.19**

**Ex6.20**

**Ex6.21**

**Deadline:NOV.29.2012**



# Principles of Automatic Control

- Chap 5 The Frequency-Response Design Method  
(5-5 steady-state analysis/5-6 dynamic analysis)

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# Chap 5 The Frequency-Response Design Method

- ◆5-1 Frequency Characteristics Definition
- ◆5-2 Nyquist Diagram
- ◆5-3 Bode Diagram
- ◆5-4 Stability Analysis
- ◆5-5 Frequency domain of steady-state analysis
- ◆5-6 Frequency domain dynamic analysis



# 5-5 Stability Margins

- ◆ We commonly used open **loop system frequency characteristics**  $G(j\omega)H(j\omega)$  close to the  $(-1, j0)$  points to represent the stability of the closed-loop system level in GH plane .
- ◆ Generally speaking,  $G(j\omega)H(j\omega)$  **leave**  $(-1, j0)$  point **more far**, is **the higher level of stability**; Conversely, the **lower level of stability**.

(The Nyquist Stability Criterion)



# 5-5 Stability Margins

- ◆ we judge the system stability with **the Routh-Hurwitz criterion (In Chap3.4) and The Nyquist Stability Criterion(In Chap5.4)**. The results show the system is **stable or unstable**.
- ◆ In really situation, the system is stable for all small gain values and becomes unstable if the gain increases past a certain critical point.



# 5-5 Stability Margins

- ◆ **Open-loop system frequency characteristics**, two commonly quantities that measure the stability margin.

Gain Margin(GM)    增益裕度

Phase Margin(PM)    相角 ( 位 ) 裕度

- ◆ **Closed-loop system frequency characteristics**, there commonly quantities that measure the stability margin.

谐振峰值 $M_r$  ; 谐振角频率 $\omega_r$  , 带宽 $\omega_b$  ,



# 5-5-1 Stability Margins- Gain Margin(GM)

## ◆ Gain Margin(GM)- $K_g$

$$K_g = \frac{1}{|G(j\omega_g)H(j\omega_g)|}$$

$K_g > 1$ , the GM is positive, the system is stable.

$K_g = 1$ , the system is critical stable.

$K_g < 1$ , the GM is negative, the system is not stable.

Take logarithm for GM with dB (**Bode plot**)

$$GM(\text{dB}) = 20\lg K_g = -20\lg |G(j\omega_g)H(j\omega_g)| \quad (\text{dB})$$

So,  $GM > 0$  dB, the system is stable.

$GM = 0$  dB, the system is critical stable.

$GM < 0$  dB, the system is not stable.





## 5-4-4 Stability Margins

- ◆ The GM is the factor by which the gain can be raised before instability results.
- ◆ For the typical case, the GM value can be read directly from the **Bode plot (See Fig6.15 in textbooks)** by measuring the vertical distance between magnitude curve and the  $|KG(j\omega)|=1$  line at the frequency where  $\angle G(j\omega) = \pi$
- ◆ The GM can also be determined from a root locus with respect to K by noting two values of K: (1) at the point where the locus crosses the  $j\omega$ -axis; (2) at the nominal closed-loop poles. The GM is the ratio of these two values.



## 5-5-2 Stability Margins- Phase Margin(PM)

### Phase Margin(PM)

cross over frequency  $\omega_c$  : the frequency corresponding to the value of open-loop magnitude is equal 1

$$|G(j\omega_c)H(j\omega_c)| = 1$$

On the crossover frequency  $\omega_c$  , the closed-loop system to achieve critical stability needed for the additional state phase shift quantity (or over ahead or late phase shift), which is defined as the Phase Margin(PM) , written for  $\gamma$  .



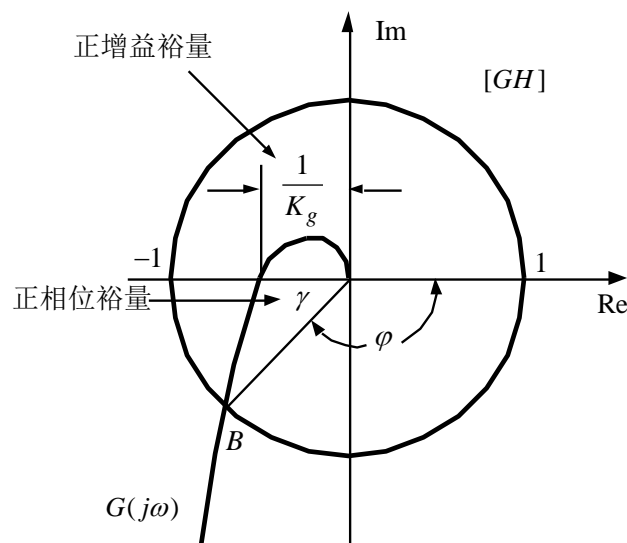
## Polar Diagram

Unit circle

5-5-1 Unit circle within round area

Unit circle outside round area

Negative real axis



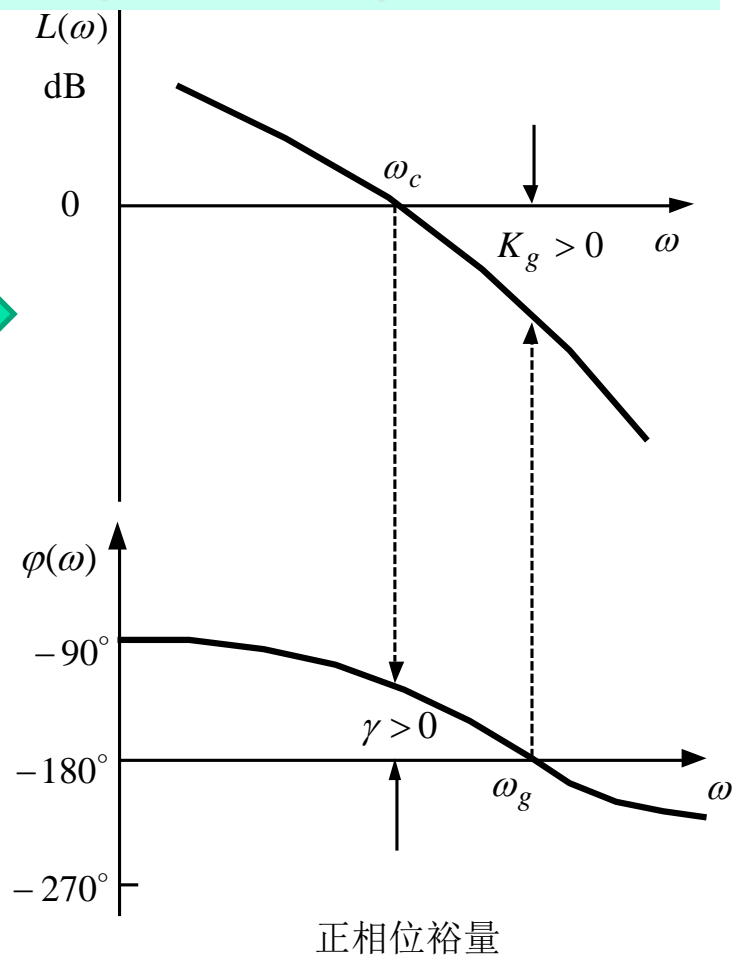
## Bode Diagram

0db line(**magnitude curves**)

0 db line the following areas

0 db line above area

-180° line(**phase curves**)

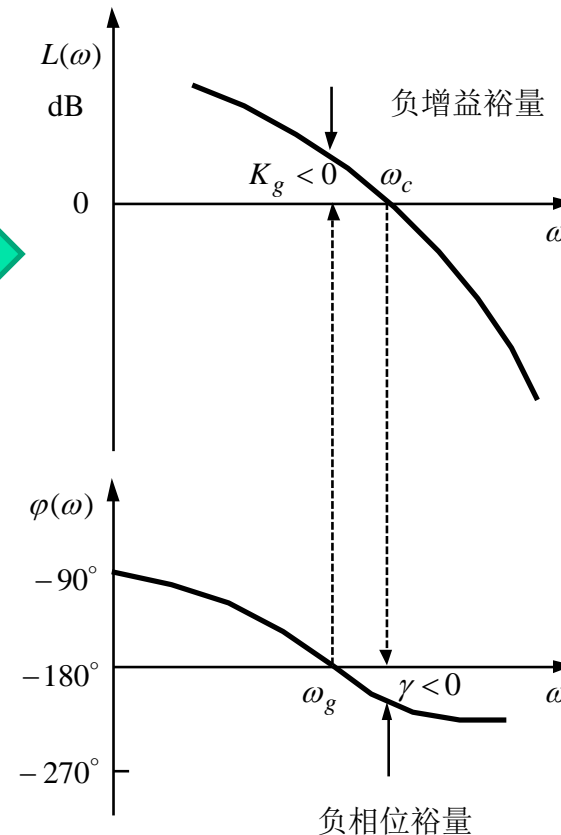
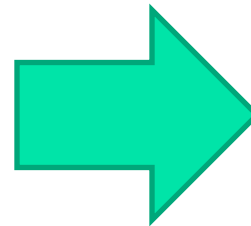
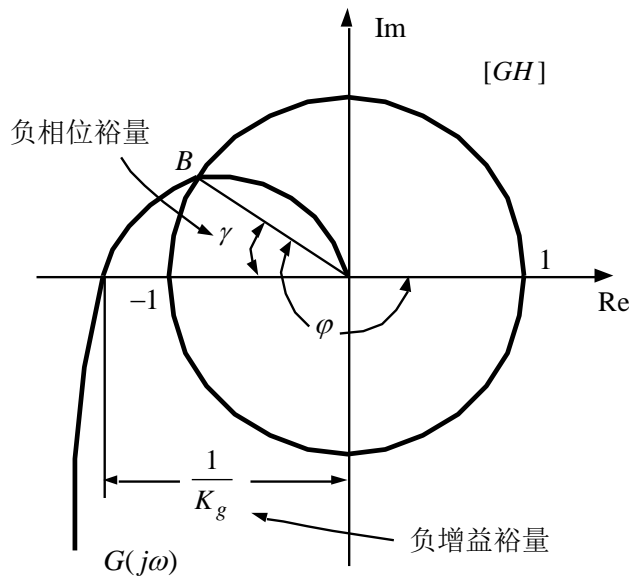


Phase margin:

when  $\gamma > 0$ , PM is negative,  
the system is stable:

$$\gamma = \varphi(\omega_c) - (-180^\circ) = 180^\circ + \varphi(\omega_c)$$

# 5-5-2 Stability Margins- Phase Margin(PM)



When  $\gamma < 0$ , PM is positive, the system is unstable.

$$\gamma = \varphi(\omega_c) - (-180^\circ) = 180^\circ + \varphi(\omega_c)$$

# 5-5-3 conclusion

- ◆  $L(\omega_c)$ , the slope for -20 db / +, the system stability.
- ◆  $L(\omega_c)$ , the slope for -40 db / +, the system may stability, also may not stable, even if stability,  $\gamma$  was very small.
- ◆  $L(\omega_c)$ , the slope for -60 db / +, the system must not stable.
- ◆ In order to make the system has a certain stability allowance, the slope of  $L(\omega)$  in  $\omega_c$  should be -20 db .



# 5-5-3 conclusion

- ◆ In general, in order to get the satisfactory performance in design system,

Phase Margin(PM) :  $30^\circ \sim 60^\circ$

Gain Margin(GM) GM:  $> 6\text{dB}$

(Experience results)



# 5-5-4 steady-state analysis

## The steady-state error

The type of system	Error coefficient $K_p$ $K_v$ $K_a$	Unit step input $r(t) = u(t)$	Unit speed input $r(t) = t$	Unit acceleration input $r(t) = \frac{1}{2}t^2$
<b>O</b>	$K$ 0   0	$\frac{1}{1+K}$	$\infty$	$\infty$
<b>I</b>	$\infty$ $K$ 0	0	$\frac{1}{K}$	$\infty$
<b>II</b>	$\infty$ $\infty$ $K$	0	0	$\frac{1}{K}$

1. The steady-state error **has relationship with** the input, the structure of the system.
2. Reduce or eliminate the steady-state error method:
  - A. **increase** the open loop amplification coefficient **K**;
  - B. improve the system type number(0-> II);



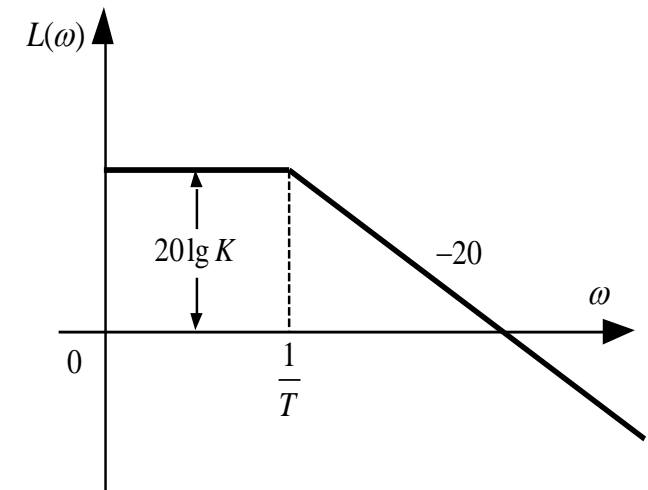
# 5-5-4 steady-state analysis

**Key: How to get steady-state error coefficient on the logarithm frequency characteristics curves**

## 1、0 type system

Set: a system of open loop frequency characteristics:

$$G(j\omega) = \frac{K}{j\omega T + 1} \quad K = K_P$$



The amplitude frequency characteristics as shown in figure. At low frequency ( $\omega \rightarrow 0$ ), the amplitude  $L(\omega) = 20K = 20\lg K_P$

**Conclusion:** 0 type system logarithm amplitude frequency characteristics curve of low frequency band slope for 0; Height is  $20\lg K_P$ , which is the system of  $K_P$  steady position error coefficient.



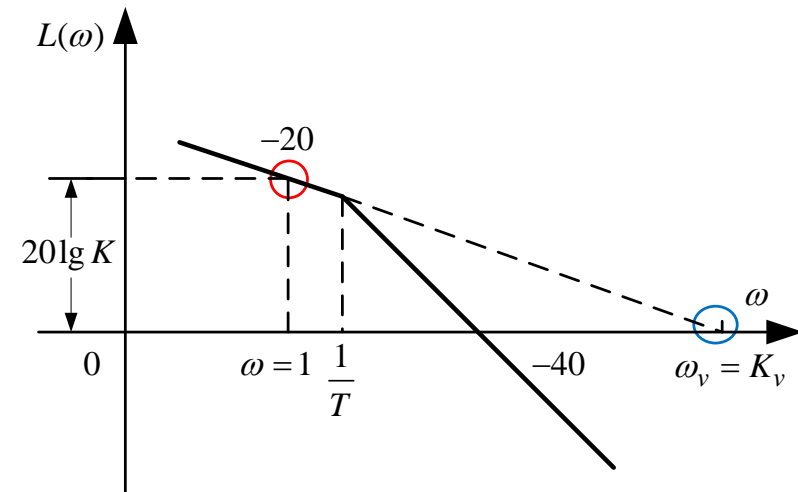


# 5-5-4 steady-state analysis

## 2、 I type system

Set: a system of open loop frequency characteristics

$$G(j\omega) = \frac{K}{j\omega(j\omega T + 1)}$$



The amplitude frequency characteristics as shown in figure.

- 1) For type I system, the logarithm of system frequency response curve with the -20 dB/dec slope in the low frequency band, and its(or its extension line) intersection point with  $\omega = 1$  line, which corresponding amplitude is  $20 \lg K_v$ , prove as follows:

When  $\omega=1$ :

$$20 \lg \left| \frac{K_v}{j\omega} \right|_{\omega=1} = 20 \lg K_v$$



# 5-5-4 steady-state analysis

2 ) the cross frequency  $\omega_v$  for -20 dB/dec slope (or its extension line) and 0 dB intersection line , its numerical value is  $K_v$ .  
Prove as follows:

$$20\lg\left|\frac{K_v}{j\omega}\right|_{\omega=\omega_1} = 0 \text{ (dB)} \quad \left|\frac{K_v}{j\omega_1}\right| = 1$$

so:  $K_v = \omega_1$



# 5-5-4 steady-state analysis

## 3. II type system

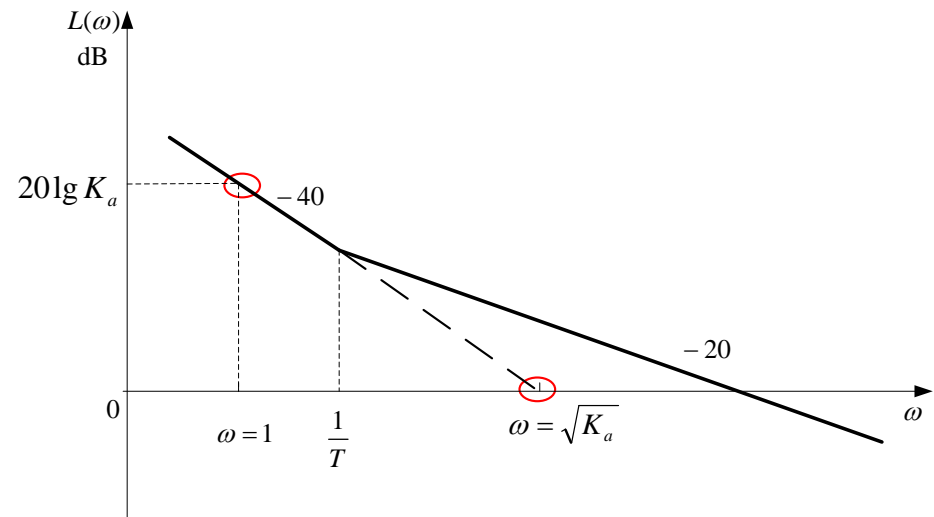
$$G(j\omega) = \frac{K(j\omega T + 1)}{(j\omega)^2}$$

For type II system, the logarithm of system frequency response curve with the -20 dB/dec slope in the low frequency band, and its (or its extension line) intersection point with  $\omega = 1$  line, which corresponding amplitude is  $20 \lg K_a$ , prove as follows:

Type II system: the low frequency band frequency characteristics

$$(\omega \ll 1) \quad G(j\omega) = \frac{K}{(j\omega)^2}$$

So, when  $\omega=1$ :  $20 \lg \left| \frac{K_a}{(j\omega)^2} \right|_{\omega=1} = 20 \lg K_a$



# 5-5-4 steady-state analysis

2 ) the cross frequency  $\omega_a$  for -40 dB/dec slope (or its extension line) and 0 dB intersection line , its numerical value is square root of  $K_a$ .

Prove: Because  $K_a = \omega_a^2$

so:  $\omega_a = \sqrt{K_a}$   $20\lg\left|\frac{K_a}{(j\omega)^2}\right|_{\omega=\omega_a} = 0 \text{ (dB)}$

- ◆ Improve the open loop system frequency characteristics of low frequency band amplitude or increase the slope of the low frequency band absolute value (model number), are **conductive to the decrease of the steady-state error of the system.**
- ◆ Conclusion: generally speaking, for the closed-loop system, open-loop frequency characteristics of **low frequency characterization express its steady characteristics.**



# 5-6 Dynamic analysis

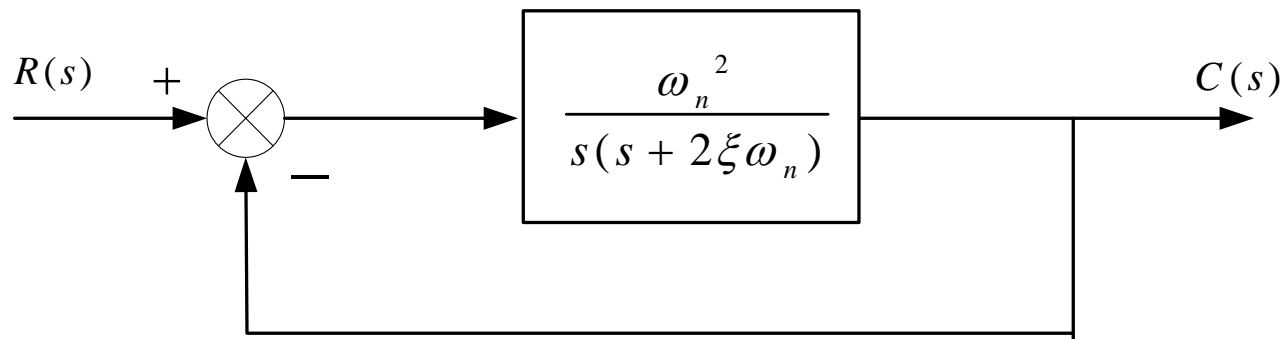
- ◆ Because human's instinct is to build in **time domain**, so, it is often proposed indexes in time domain in real project .
- ◆ Studies show that, for **second order systems** , for those index , there are strict mathematical relationship between the time domain and frequency domain. For the **high order system**, this relationship is more complex, often used in engineering approximation formula or curve to express the relationship.



# 5-6 Dynamic analysis

- 1、 the relationship between the dynamic performance index :  $\sigma_p$  in time domain and the open loop indexes :  $\omega_c$  in frequency domain .

## (1) $\gamma$ order system



# 5-6 dynamic analysis

( 1 ) the relationship between  $\sigma_p$  and  $\gamma$   
 Open-loop transfer function:  $G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$

**Magnitude characteristics :**  $|G(j\omega)| = \frac{\omega_n^2}{\sqrt{(-\omega^2)^2 + (2\zeta\omega_n\omega)^2}}$

Set  $|G(j\omega_c)| = 1$  ,SO  $\omega_c = \omega_n \sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2}$   $\omega_n = \frac{\omega_c}{\sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2}}$

So the Phase Margin(PM) :

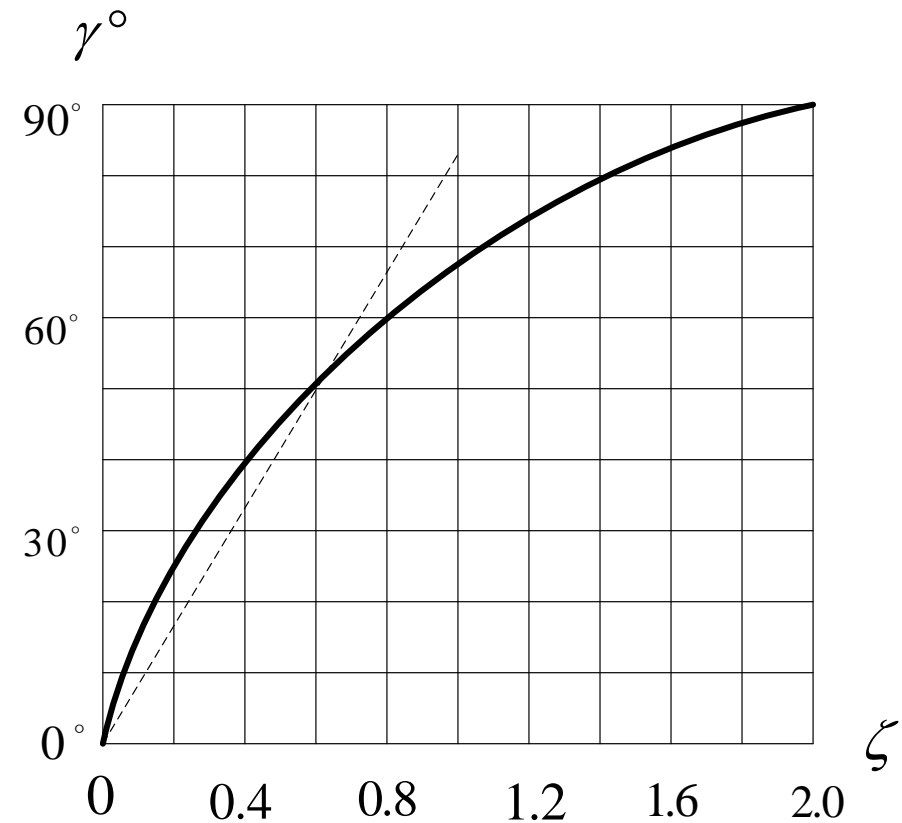
$$\gamma = 180^\circ + \angle G(j\omega_c) = 90^\circ - \operatorname{tg}^{-1} \frac{\sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2}}{2\zeta} = \operatorname{tg}^{-1} \frac{2\zeta}{\sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2}}$$



# 5-6 dynamic analysis

- ◆ When  $0 < \gamma < 60^\circ$ , they are approximate linear relationship for  $\gamma$  and  $\zeta$ ;  $\zeta$  is shown as the dotted line in the diagram, at this time:

$$\gamma = 100\zeta$$



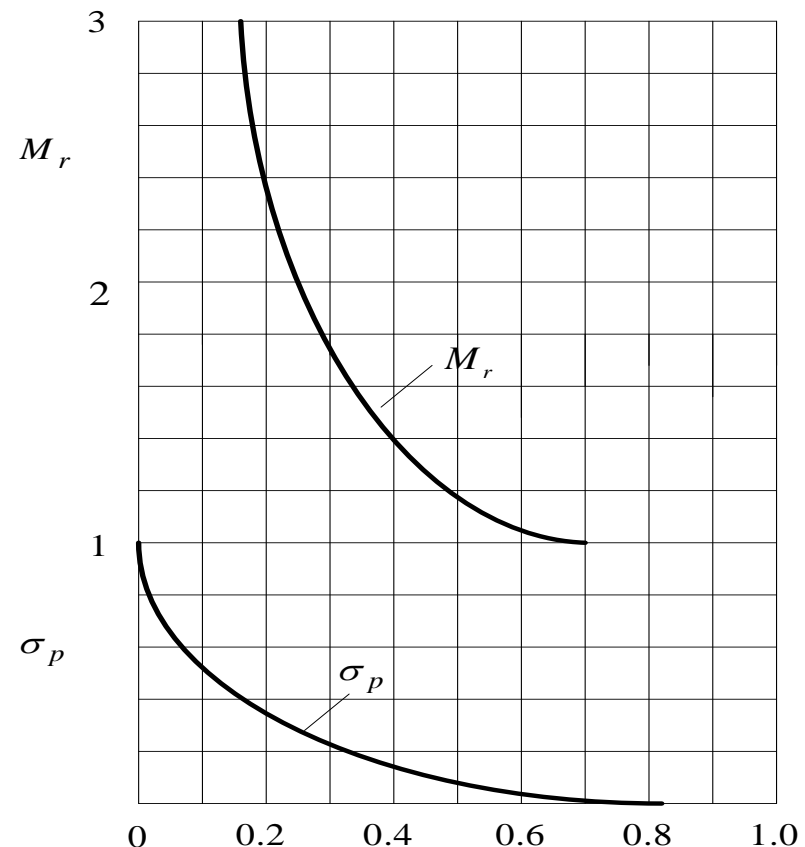


# 5-6 dynamic analysis

The time domain analysis shows that the second order systems maximum overshoot:

$$\sigma_p = e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

It is through the middle parameters  $\zeta$  to associated with the  $\gamma$  and  $\sigma_p$ .



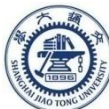
# 5-6 dynamic analysis

## Conclusion :

For second order systems : The smaller  $\gamma$ , the bigger  $\sigma_p$  the bigger, the smaller.

To make the second order systems not oscillation too far and adjusting the time is not too long, generally :

$$30^\circ \leq \gamma \leq 70^\circ$$



# 5-6 dynamic analysis

( 2 ) the relationship between

$$t_s \quad \text{and} \quad \omega_c \quad \gamma$$

Because :

$$t_s \approx \frac{3}{\zeta \omega_n}$$

and  $\omega_n = \frac{\omega_c}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}$

$$t_s \omega_c = \frac{3\sqrt{\sqrt{1+4\zeta^2} - 2\zeta^2}}{\zeta}$$

So , the parameters  $\zeta$  is fixed or defined, the bigger of crossover frequency  $\omega_c$ , the more short time for transition process , and they have an **inverse correlation relationship**.



# 5-6 dynamic analysis

## 2、 High order system

For high order system, it is hard to find accurate indicator equation for the open loop frequency domain and time domain index. Introduce the following two experience formula:

$$\sigma_p \approx 0.16 + 0.4\left(\frac{1}{\sin \gamma} - 1\right) \quad (35^\circ \leq \gamma \leq 90^\circ) \quad t_s \approx \frac{K\pi}{\omega_c} \text{ (s)}$$

$$\text{where } K = 2 + 1.5\left(\frac{1}{\sin \gamma} - 1\right) + 2.5\left(\frac{1}{\sin \gamma} - 1\right)^2 \quad (35^\circ \leq \gamma \leq 90^\circ)$$

so, we can get:

high order system :  $\sigma_p$  随着  $M_r$  增大而增大。  
过渡过程时间  $t_s$  随  $M_r$  增大而增加，随  $\omega_c$  增大而减小。



# 5-6 dynamic analysis

## conclusion

Based on the analysis for second order systems and high order systems above, it shows that two **important parameters**  $\gamma$  and  $\omega_c$  for the open loop system frequency characteristics. They reflect response characteristics of the closed-loop system in time domain.

In other words: closed loop system **dynamic performance** depends mainly on the intermediate frequency of the open loop logarithm characteristics.



# 5-6 dynamic analysis

**Ex :**

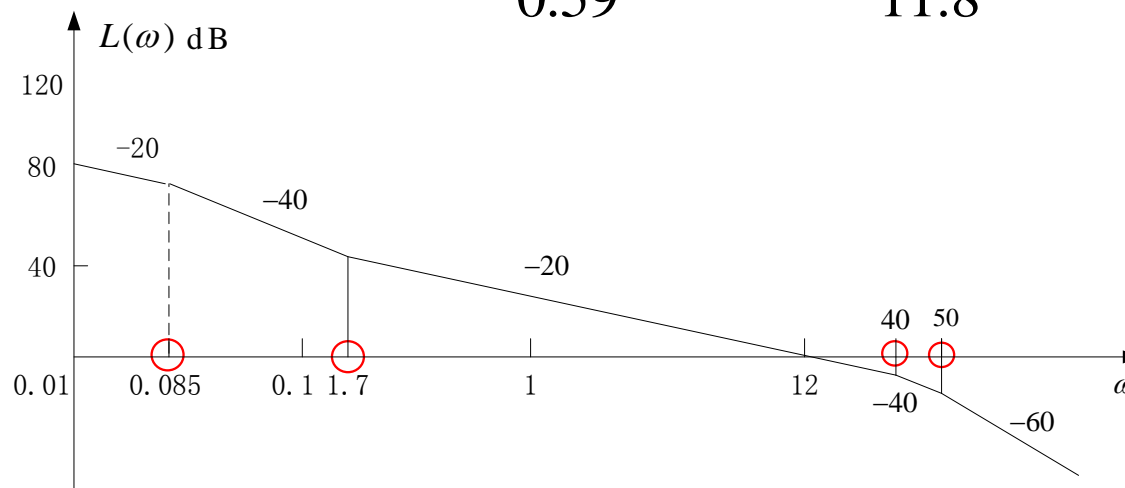
$$G(s) = \frac{250(0.59s + 1)}{s(0.02s + 1)(0.025s + 1)(11.8s + 1)}$$

Try estimation performance index .

**S : Open-loop amplification coefficient  $K=250$**   
 $20\lg K = 20\lg 250 = 48(\text{db})$

The cross frequency for each element:

$$\frac{1}{0.02} = 50 \quad \frac{1}{0.025} = 40 \quad \frac{1}{0.59} = 1.7 \quad \frac{1}{11.8} = 0.085$$



# 5-6 dynamic analysis

In this Figure  $\omega_c = 12\text{s}^{-1}$ , it is a minimum-phase system, so the PM :

$$\begin{aligned}\gamma &= 180^\circ + \varphi(\omega_c) = 180^\circ + (-\text{tg}^{-1}0.02 \times 12 - \text{tg}^{-1}0.025 \times 12 \\ &\quad - \text{tg}^{-1}11.8 \times 12 + \text{tg}^{-1}0.59 \times 12 - 90^\circ) \\ &= 180^\circ + (-13.5^\circ - 16.7^\circ - 89.6^\circ + 82^\circ - 90^\circ) = 52.2^\circ\end{aligned}$$

the overshoot  $\sigma_p$  of closed-loop system and transient time  $t_s$  :

$$\sigma_p = 0.16 + 0.4\left(\frac{1}{\sin 52^\circ} - 1\right) = 0.16 + 0.108 = 0.27 = 27\%$$

$$k = 2 + 1.5\left(\frac{1}{\sin 52^\circ} - 1\right) + 2.5\left(\frac{1}{\sin 52^\circ} - 1\right)^2 = 2 + 1.5 \times 0.27 + 2.5 \times (0.27)^2 = 2.587$$

$$t_s = \frac{k\pi}{\omega_c} = \frac{2.587 \times 3.14}{12} = 0.68(\text{s})$$



# 5-6 dynamic analysis

Closed loop frequency domain index:

(1) zero frequency amplitude  $M_0$ :

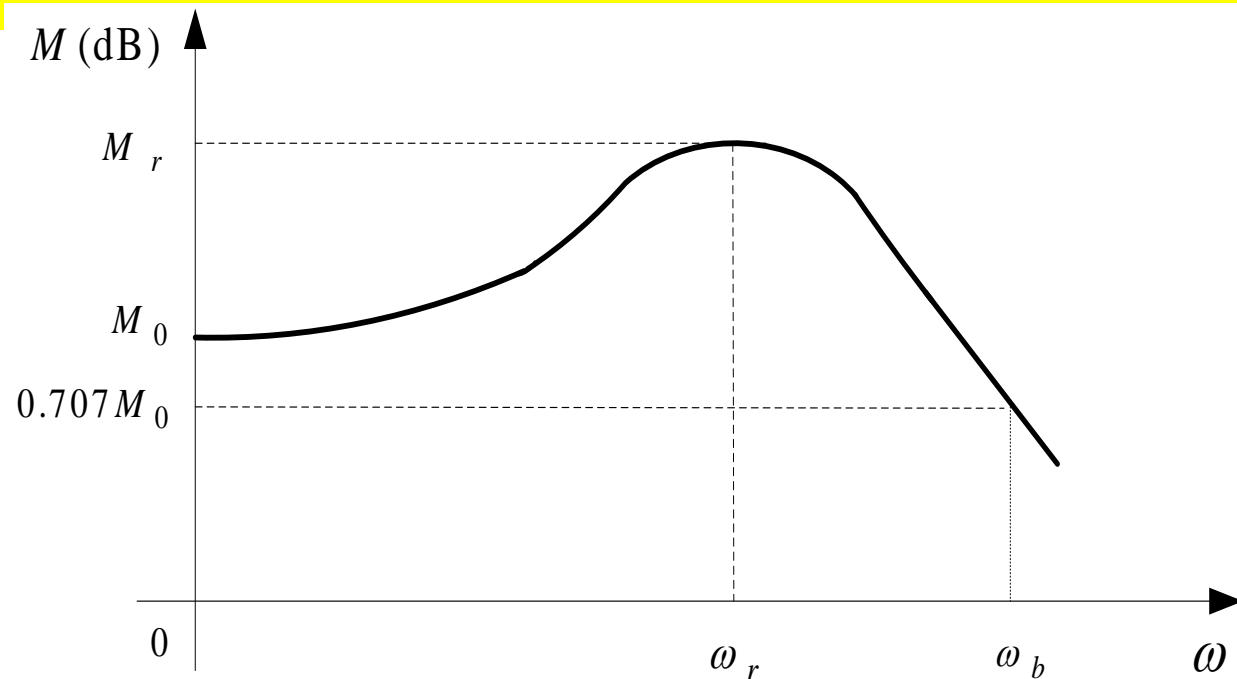
$\omega = 0$  closed-loop amplitude.

(2) the resonance peak  $M_r$ :

Closed loop amplitude frequency maximum.

(3) the resonant frequency  $\omega_r$ :

Resonance when peak frequency.



(4) the system bandwidth  $\omega_b$ :

Closed loop amplitude reduced to 0.707 of the frequency  $M_0$ . called system bandwidth.

(5) shear rate: the slope of  $M(\omega)$  at  $\omega_b$ . The system can reflect the anti-jamming, **slope is bigger, the greater the anti-interference ability is stronger.**

Usually use  $M_r$  and  $\omega_b$  (or  $\omega_r$ ) as the closed-loop system time-frequency dynamic index





# 5-6 dynamic analysis

## 1. Two order system

### 1) $\sigma_p/M_r$

The resonance frequency:

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \quad (0 \leq \zeta \leq 0.707)$$

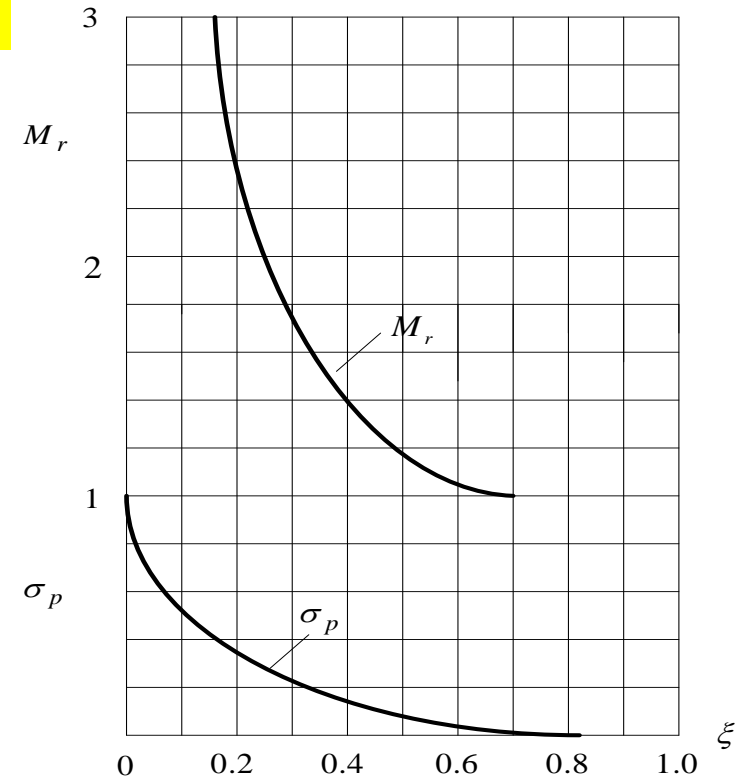
The resonance peak:

$$M_r = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \quad (0 \leq \zeta \leq 0.707)$$

$M_r \propto \zeta$  (成反比)。

$\zeta$  is fixed,  $M_r \uparrow$ ,  $\sigma_p \uparrow$ , The convergent slower, stability and efficiency are poor.

When  $M_r = 1.2 \sim 1.5$ ,  $\sigma_p = 20 \sim 30\%$ , then can get moderate oscillation properties. If appear  $M_r > 2$ , then the corresponding  $\sigma_p$  can be as high as 40% above.



# 5-6 dynamic analysis

## 2) $t_s/\omega_b$

In  $\omega_b$ , the system amplitude frequency:

$$M(\omega_b) = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta \omega_n \omega_b)^2}} = 0.707$$

Can get the relationship between  $\omega_b$ 、 $\omega_n$ 、 $\zeta$

$$\omega_b = \omega_n \sqrt{1 - 2\zeta^2} + \sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

$$t_s = \frac{3}{\zeta \omega_n} \quad \omega_b t_s \approx \frac{3}{\zeta} \sqrt{1 - 2\zeta^2} + \sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

so when  $\zeta$  is fixed,  $t_s$   $\omega_b$  (成反比)。



# 5-6 dynamic analysis

## 2. High order system

For high order system, difficult to find out the exact relationship. it shown that , when  $\gamma$  is small, the experience formula is always used.

$$(M_r = \frac{1}{\sin \gamma})$$

Closed loop frequency domain indexes and can be expressed as form:  $\sigma_p = 0.16 + 0.4(M_r - 1)$  ( $1 \leq M_r \leq 1.8$ )

and  $t_s = \frac{Kp}{\omega_c} (s)$

Where:  $K = 2 + 1.5(M_r - 1) + 2.5(M_r - 1)^2$  ( $1 \leq M_r \leq 1.8$ )

**High order system :**  $M_r$   $\rightarrow$   $\sigma_p$   
 for  $M_r$   $t_s$   $\rightarrow$



◆ Homework

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**Ex6.23**

**Ex6.24**

Deadline:DEC.5.2012



# Schedule for following time

- 12<sup>th</sup> week (Nov.26,28) Chap5.5/5.6
- **13<sup>th</sup> week (Dec.3,5) Course Design Chap5**
- 14<sup>th</sup> week (Dec.10,12) Chap6.1~6.4
- 15<sup>th</sup> week (Dec.17,19) Course Design Chap6
- 16<sup>th</sup> week (Dec.24) General Review for ACT
- 16<sup>th</sup> week (Dec.27) Final Exam

