

Principles of Automatic Control

-Chapter 2 Dynamic Models and Typical Elements

School of Aeronautics and Astronautics

- Assoc. Prof. Xiao Gang
- Email: xiaogang@sjtu.edu.cn
- Tel: 021-34206192
- Mobile: 13918459696
- Office: 1-431 Room

Chapter 2 Dynamic Models and Typical Elements

- **Mathematical Model** is the mathematical abstraction of a physical system.
- **Dynamic Model**- Mathematical Description for A Controlled Process
- **General understanding**: reveals control system variables and the relation between **internal relations analytical formula or graphics**

Chapter 2 Dynamic Models and Typical Elements

Typical Dynamic Model Examples

- Dynamic of Mechanical System
- Models of Electric Circuits
- Models of Electromechanical System
- Heat and Fluid-Flow Models
- Complex Mechanical Systems

Mathematical modeling of the system to be controlled is the **First Step** in analyzing and designing the required system controls

Chapter 2 Dynamic Models and Typical Elements

—Why do we must study the model of control system?

An accurate mathematic model that describes a system completely must be determined in order to analyze and control a dynamic system.

Chapter 2 Dynamic Models and Typical Elements

The Steps of Analyzing and Studying a Dynamic System

1. Define the system and its components.
2. Formulate the mathematic model and list the necessary assumptions.
3. Write the differential equations describing the model.
4. Solve the equations for the desired output variables.
5. Examine the solutions and the assumptions.
6. If necessary, reanalyze or redesign the system

Chapter 2 Dynamic Models and Typical Elements

1. Mathematical Models of Systems

- **Graph models- 图模型**
 - Block diagram
 - Signal-flow graph
- **Mathematical models- 数学模型**
 - Differential equation
 - Transfer function
 - Frequency response
- **Literal models- 文字模型**
 - Arithmetic language

Chapter 2 Dynamic Models and Typical Elements

- 1. Mathematical Models of Systems
- Model have different features depends on what characteristics do you want to study
 - If analysis research the dynamic behavior of the system, take its mathematical model is convenient
 - If research the internal structure of system analysis, take its physical model is better
 - If both, then take the figure model is reasonable

Chapter 2 Dynamic Models and Typical Elements

2. “Tri-domain” models and mutual relations

Differential equation

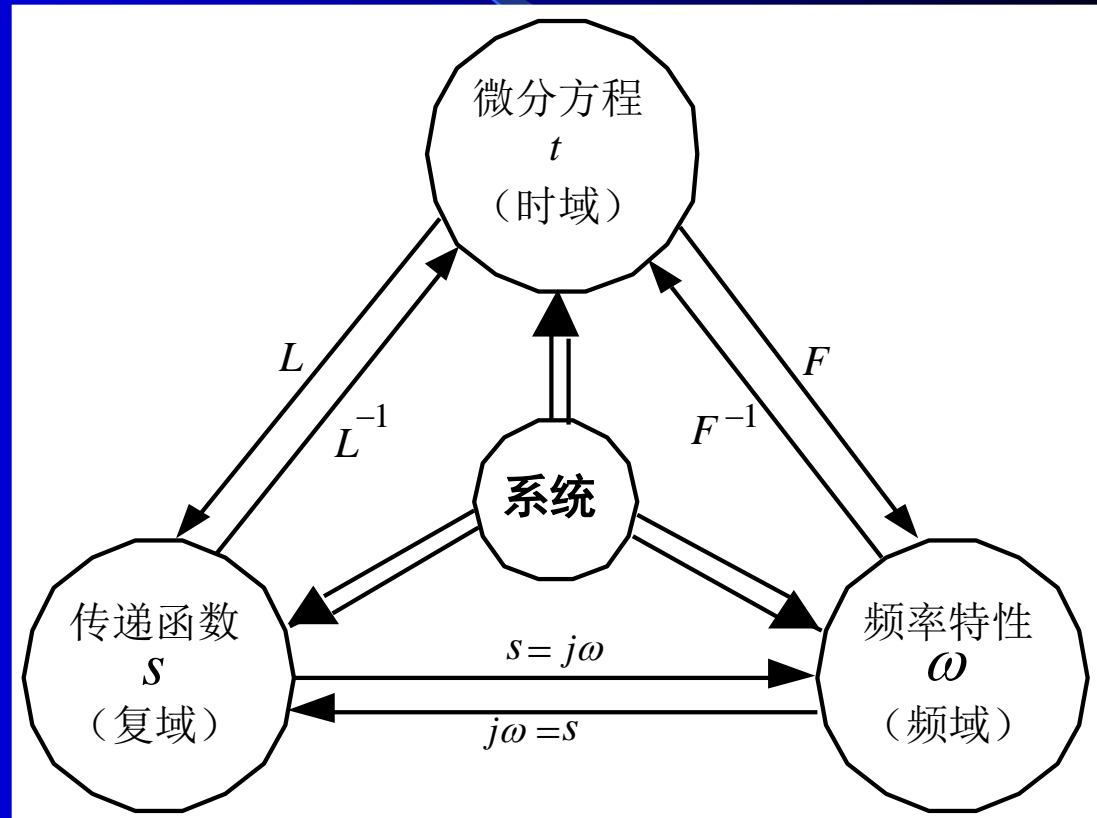
(time domain)

Transfer function

(complex frequency domain)

Frequency response

(frequency domain)



Chapter 2 Dynamic Models and Typical Elements

E.g. Establish RC circuit dynamic equation

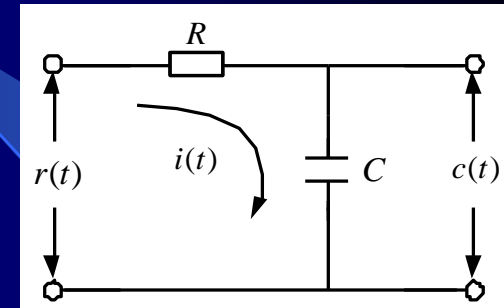
$r(t)$ — Input

$c(t)$ — Output

Time domain :

$$T \frac{dC(t)}{dt} + C(t) = r(t)$$

$$RC = T$$



—— Differential equation

Complex frequency domain:

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{Ts + 1}$$

—— Transfer function

Frequency domain:

$$G(j\omega) = \frac{\dot{c}}{\dot{r}} = \frac{1}{RCj\omega + 1} = \frac{1}{jT\omega + 1}$$

—— Frequency response

Chapter 2 Dynamic Models and Typical Elements

- **Differential equation, transfer function and frequency characteristics** are **system mathematical models** respectively in **The time domain, the complex domain, frequency domain**
- People in research and analysis the characteristics of a control system, can according to the characteristics of the object, and set up different mathematical model of the field based on the need of project and artificially
 - **mathematical analysis:** differential equation, analysis system
 - **Engineering analysis:** transfer function, frequency characteristics, analysis

Chapter 2 Dynamic Models and Typical Elements

Generally speaking, **engineering analysis** is more **intuitive and convenient** than mathematical analysis method.

Thus, this is reason to introduce the **complex domain** and **frequency domain** of the main mathematical model.

Laplace Transform (Review)

Properties of Laplace transform

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt,$$

A. Conditions for the existence of $F(s)$

- (1) $f(t) = 0, \quad t < 0;$
- (2) $f(t)$ is integrable for any interval $t \in [a, b]$
- (3) there are constants $M > 0$ and $s_0 > 0$, for which $|f(t)| \leq Me^{s_0 t}$, for any t .

B. Linearity

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

Laplace Transform (Review)

C. Delay theorem

$$\mathcal{L}(f(t - \tau)) = e^{-s\tau} F(s)$$

D. Shifting theorem

$$\mathcal{L}(e^{-at} f(t)) = F(s + a)$$

E. Transform of derivatives

$$\mathcal{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

F. Final value theorem

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

G. The relationship between time and frequency

$$G. \quad \mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$$

Important Definition

- Transfer function:

A transfer function is a mathematical representation, in terms of spatial or temporal frequency, of the relation between the input and output of a linear time-invariant

In its simplest form for continuous-time input signal $x(t)$ and output $y(t)$, the transfer function is the linear mapping of the Laplace transform of the input, $X(s)$ to the output $Y(s)$: $Y(s) = H(s) X(s)$

or

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{x(t)\}}$$

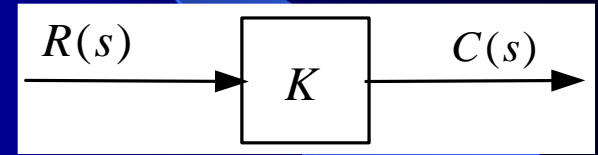
where $H(s)$ is the transfer function of the linear, time-invariant systems.

Chap 2 Dynamic Models and Typical Elements

1、Proportion Element (or Amplifying Element) (1、比例环节, 或放大环节)

Features: according to a certain proportion, Output depend on input, no lag and no distortion.

Dynamic equation : $c(t)=Kr(t)$

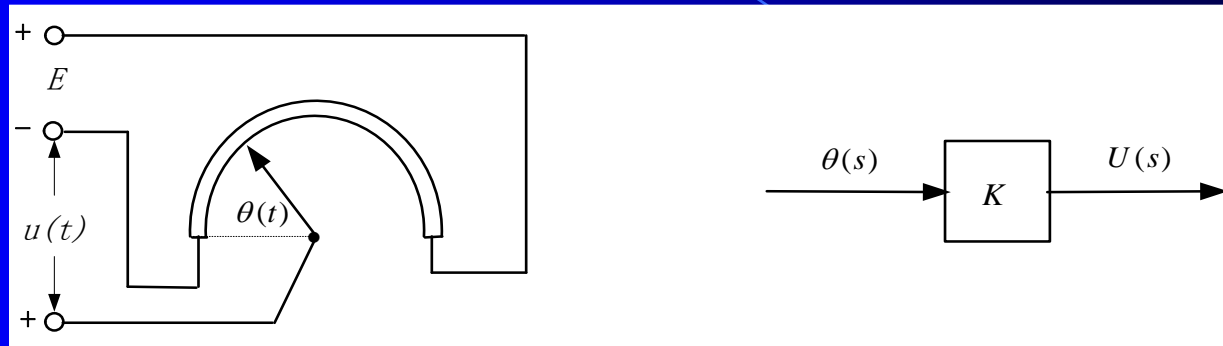


K——Amplification coefficient, are usually have dimension

Transfer function: $H(s) = \frac{C(s)}{R(s)} = K$

Frequency response: $H(j\omega) = \frac{C(j\omega)}{R(j\omega)} = K$

EX1: Input: $\theta(t)$ ——angle E ——constant voltage
 Output: $u(t)$ ——voltage



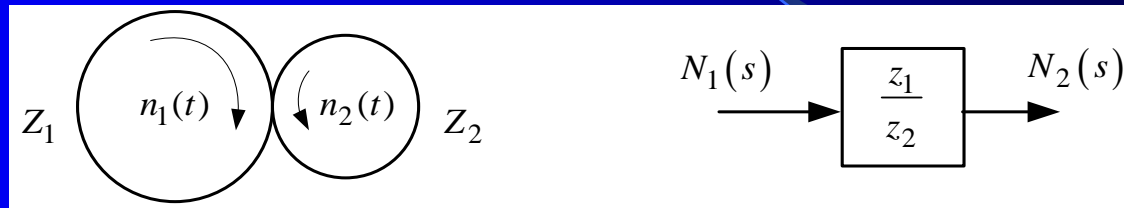
Dynamic equation : $u(t) = K\theta(t)$

Transfer function:
$$H(s) = \frac{U(s)}{\theta(s)} = K$$

K ——proportion coefficient, v/rad as dimension。

Frequency response: $H(j\omega) = K$

EX 2: **Input:** $n_1(t)$ ——rotate speed Z_1 ——Active wheel
Output: $n_2(t)$ ——rotate speed Z_2 —— engaged wheel



Dynamic equation :

$$n_2(t) = \frac{Z_1}{Z_2} n_1(t)$$

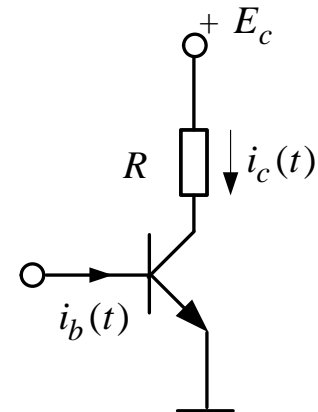
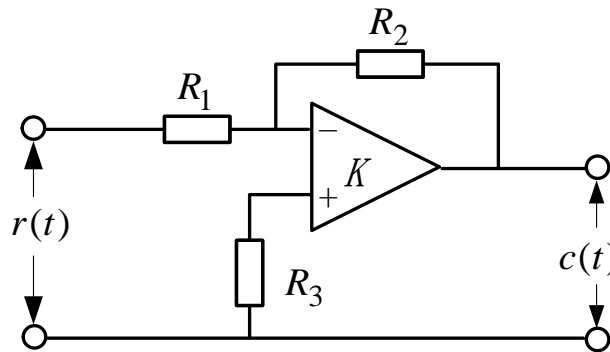
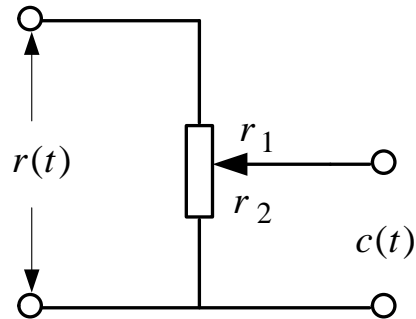
Transfer function:

$$H(s) = \frac{N_2(s)}{N_1(s)} = \frac{z_1}{z_2} = K$$

Frequency response:

$$H(j\omega) = \frac{N_2(j\omega)}{N_1(j\omega)} = \frac{z_1}{z_2} = K$$

Other Proportion Elements



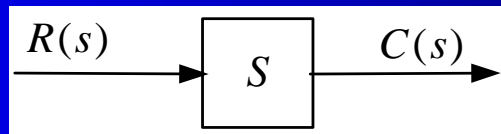
Please write Transfer function for these systems

Transfer function?

2. Differential Element

(2.微分环节)

Features: Dynamic process, the output is proportional to the input of the rate of change.



Equation of motion:

$$C(t) = K \frac{dr(t)}{dt}$$

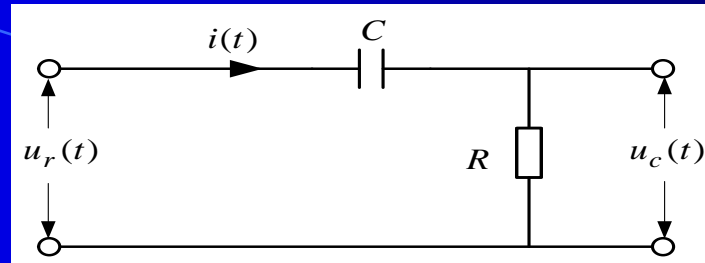
Transfer function:

$$H(s) = \frac{C(s)}{R(s)} = KS$$

Frequency response:

$$H(j\omega) = \frac{C(j\omega)}{R(j\omega)} = jK\omega$$

EX1 RC circuit



Input—— $u_r(t)$

Output—— $u_c(t)$

$$u_r(t) = \frac{1}{C} \int i(t) dt + i(t)R$$

$$i(t) = \frac{u_c(t)}{R}$$

Equation of motion:

$$u_r(t) = \frac{1}{RC} \int u_c(t) dt + u_c(t)$$

Transfer function:

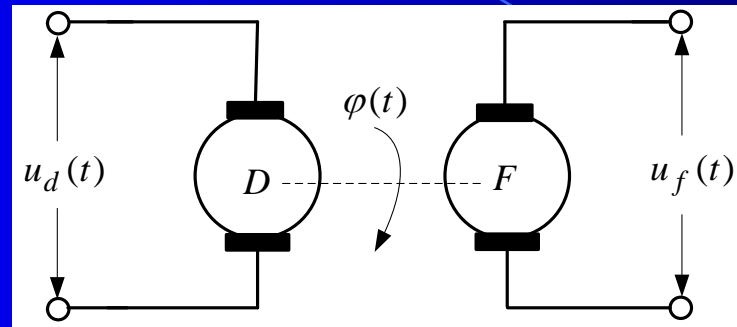
$$H(s) = \frac{U_c(s)}{U_r(s)} = \frac{T_c s}{T_c s + 1} \quad (T_c = RC)$$

When $T_c \ll 1$, the up formula is change to:

Frequency response: $G(j\omega) = jT_c\omega$

$$H(s) = \frac{U_c(s)}{U_r(s)} = T_c s$$

EX2: Mathematical Model of Tachogenerator



Input: $\varphi(t)$ ——angle of the rotor of motor D

Output: $u_f(t)$ ——armature voltage of tachogenerator F

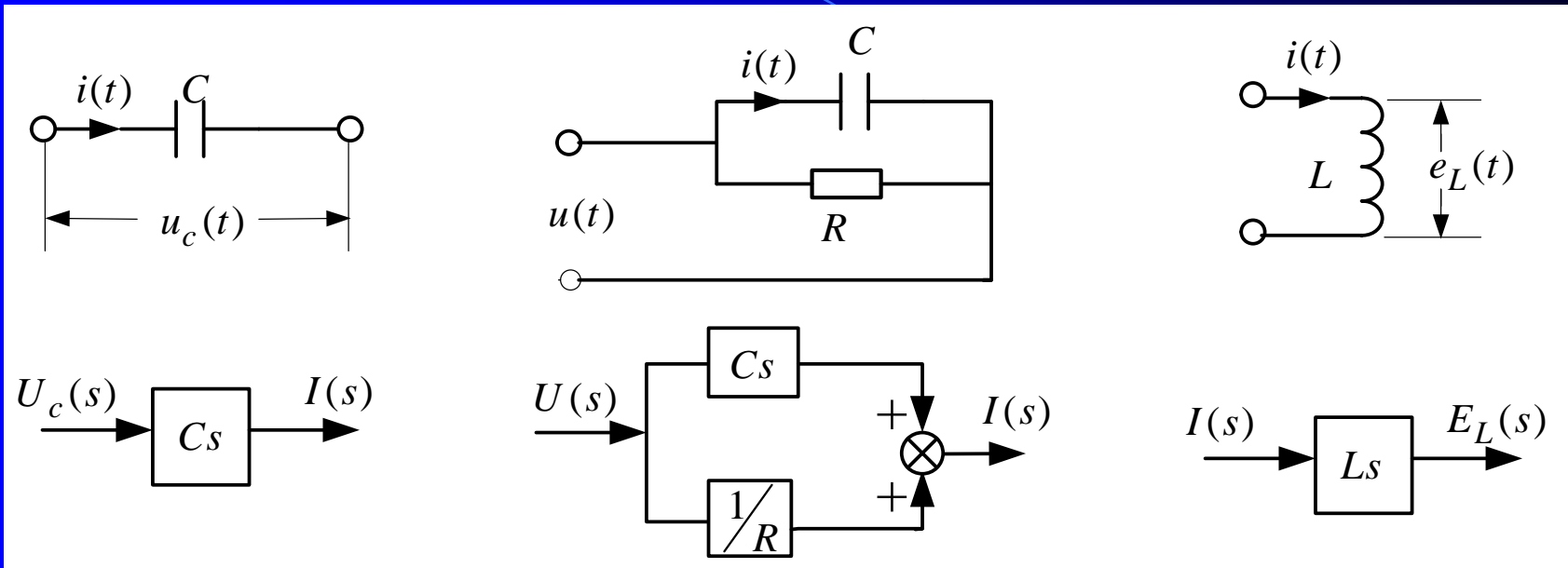
Dynamic equation:

$$u_f(t) = K \frac{d\varphi(t)}{dt}$$

Transfer function: $H(s) = Ks$

Frequency response : $H(j\omega) = jK\omega$

Other Differential Elements



Transfer function

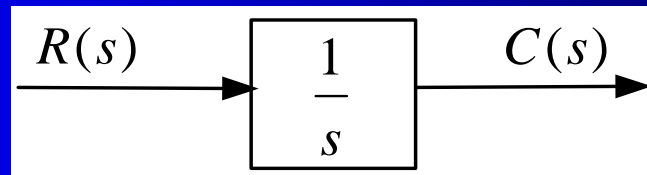
$$H(s) = \frac{I(s)}{U(s)} = Cs$$

$$H(s) = \frac{I(s)}{U(s)} = Cs + \frac{1}{R}$$

$$H(s) = \frac{E(s)}{I(s)} = Ls$$

3、Integral Element (积分环节)

Feature: The change rate of output is proportional to the input amount.



Dynamic equation:
$$\frac{dc(t)}{dt} = Kr(t)$$

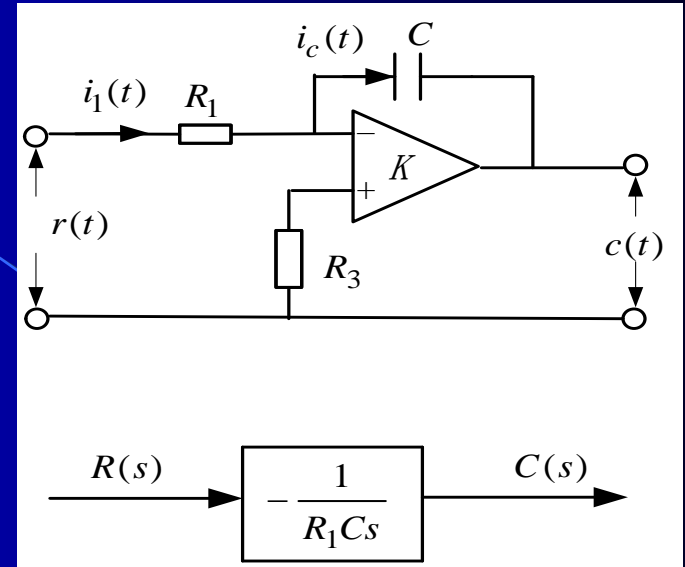
Transfer function:
$$H(s) = \frac{K}{s}$$

Frequency Response:
$$H(j\omega) = \frac{K}{j\omega}$$

EX1: Integral circuit

Input : $r(t)$, Output: $c(t)$

$$i_c(t) = i_1(t) = \frac{r(t)}{R_1}$$



Dynamic equation:

$$c(t) = -\frac{1}{C} \int i_c(t) dt = -\frac{1}{R_1 C} \int r(t) dt = -\frac{1}{T} \int r(t) dt$$

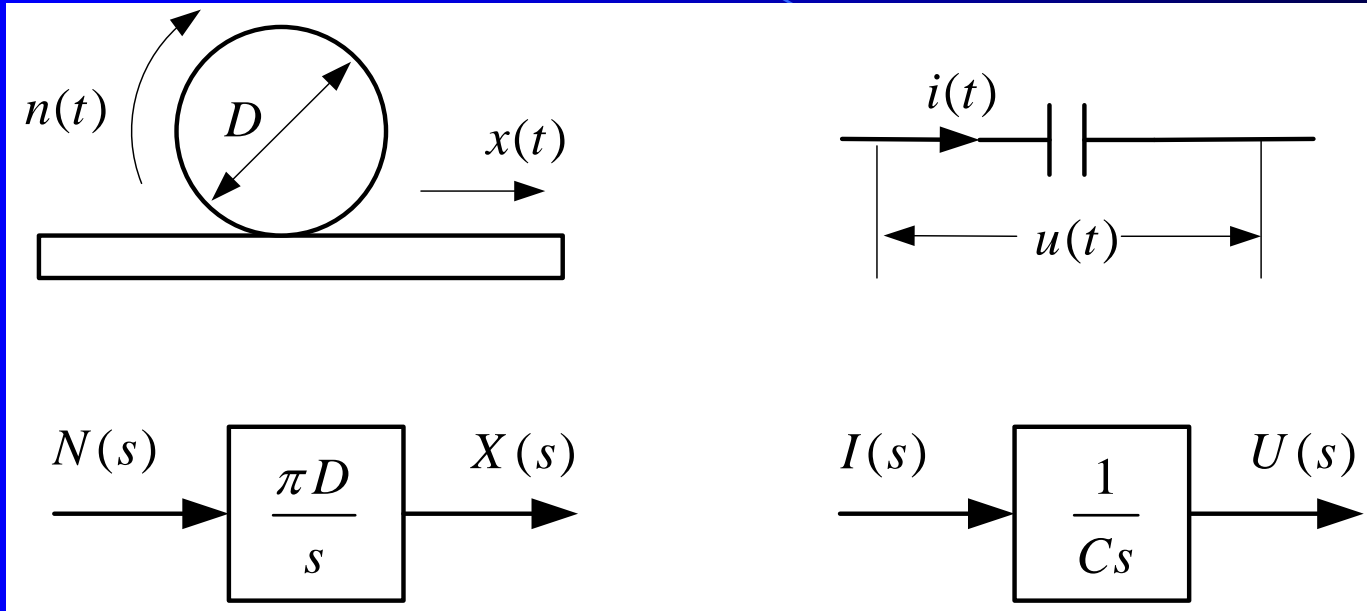
Transfer function:

$$G(s) = \frac{C(s)}{R(s)} = -\frac{1}{Ts} = -\frac{K}{s} \quad (T=R_1 C)$$

Frequency function:

$$G(j\omega) = \frac{C(j\omega)}{R(j\omega)} = -\frac{K}{jT\omega}$$

Other Integral Elements



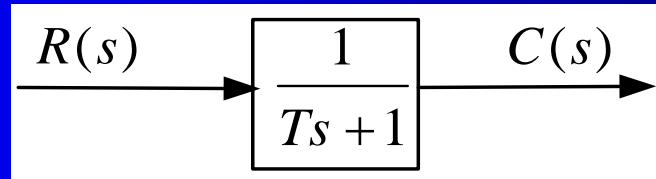
Transfer function

$$H(s) = \frac{X(s)}{N(s)} = \frac{\pi D}{s}$$

$$H(s) = \frac{U(s)}{I(s)} = \frac{1}{Cs}$$

4、Inertial Element(or Non-periodic Element) (4、惯性环节，或非周期环节)

Features: This link has an independent energy storage components, so that the input of mutations, it can't immediately reiteration, existing output of the delay in time.



Dynamic equation:

$$T \frac{dc(t)}{dt} + c(t) = Kr(t)$$

Transfer function:

$$G(s) = \frac{K}{Ts + 1}$$

Frequency response:

$$G(j\omega) = \frac{K}{jT\omega + 1}$$

EX1: DC Motor

Input: u_d ——armature voltage

Output: i_d ——armature current

Dynamic equation:

$$L_d \frac{d}{dt} i_d + R_d i_d = u_d$$

$$\tau_d = \frac{L_d}{R_d}$$

$$\tau_d \frac{d}{dt} i_d + i_d = \frac{u_d}{R_d}$$

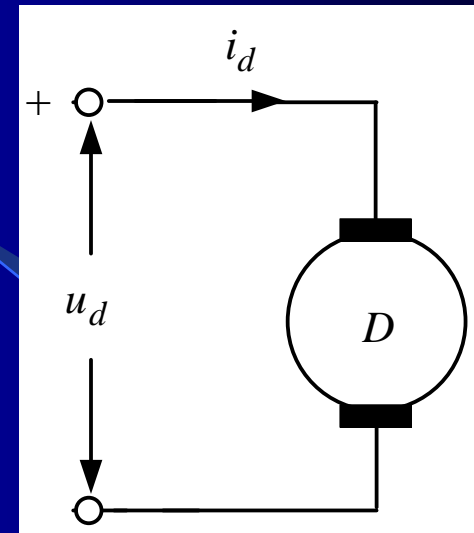
Transfer function:

$$H(s) = \frac{I_d(s)}{U_d(s)} = \frac{\frac{1}{R_d}}{\tau_d s + 1}$$

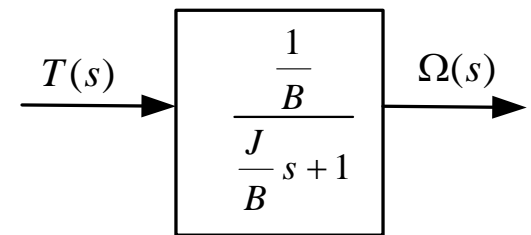
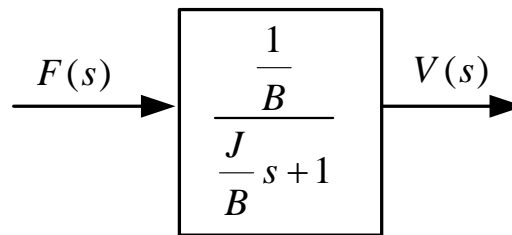
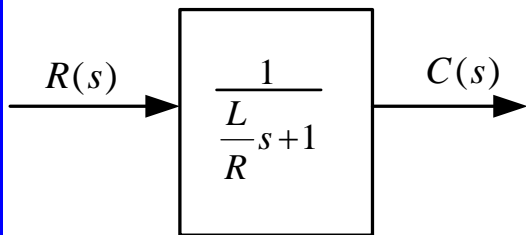
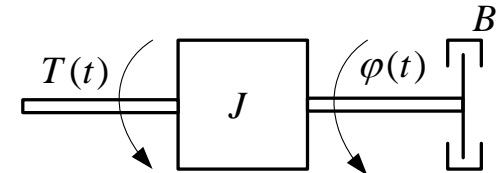
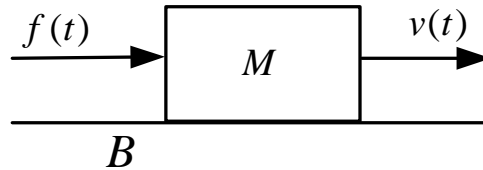
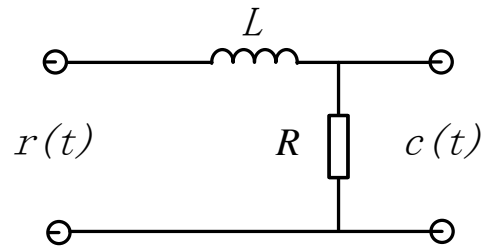
L_d ——armature inductance;

R_d ——armature resistance;

τ_d ——armature time constant;

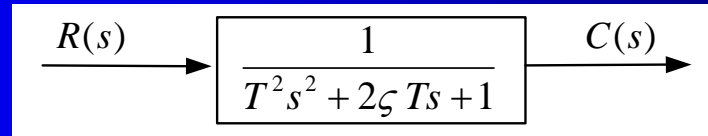


Other inertial elements



5. Oscillation Element (振荡环节)

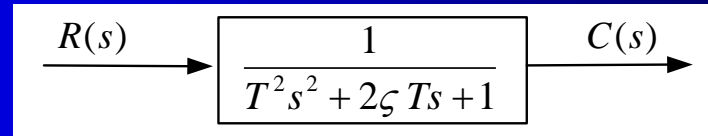
Features: Contains two independent energy storage components, when the input of the change, two energy storage of components of the energy exchange, to make the output with the nature of the oscillation.



Dynamic equation:

$$T^2 \frac{d^2 c(t)}{dt^2} + 2\zeta T \frac{dc(t)}{dt} + c(t) = K r(t)$$

Transfer function:

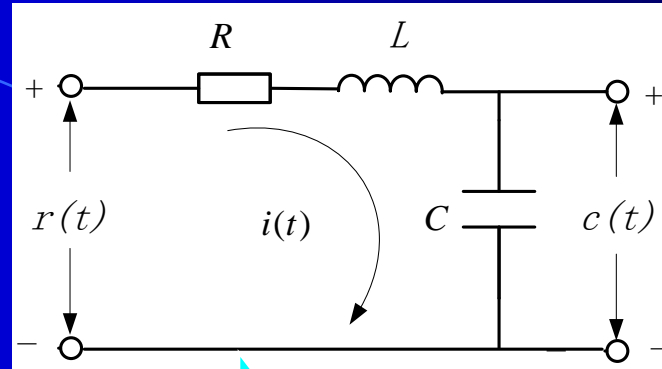


where : ζ —Damping ratio, T —Time constant

Frequency response:

$$G(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{1}{(1 - T^2 \omega^2) + j2\zeta T \omega}$$

EX1: RLC circuit



$$r(t) = L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int i(t) dt$$
$$c(t) = \frac{1}{C} \int i(t) dt$$

Elimination
Variables
 $i(t)$

$$LC \frac{d^2 c(t)}{dt^2} + RC \frac{dc(t)}{dt} + c(t) = r(t)$$

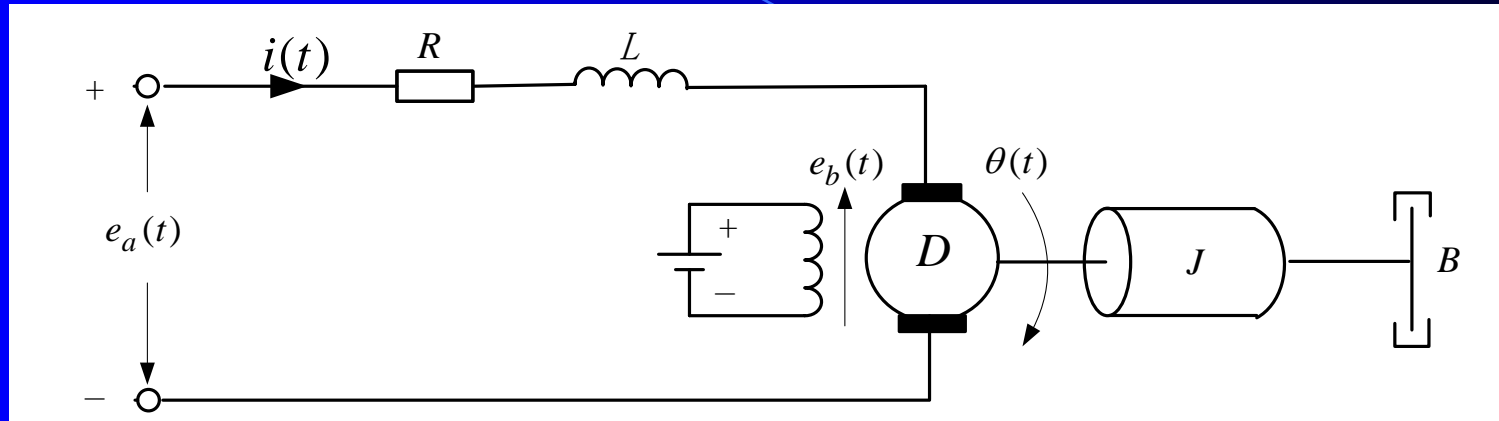
Transfer function:

$$G(s) = \frac{1}{LCs^2 + RCs + 1}$$

Frequency response:

$$G(j\omega) = \frac{1}{LC(j\omega)^2 + RC(j\omega) + 1} = \frac{1}{(1 - LC\omega^2) + jRC\omega}$$

EX2 Armature-controlled DC Motor



$e_a(t)$ --- Input voltage applied to the armature

$\theta(t)$ ---Output : angular displacement of the shaft

R -----Armature resistance;

L ----- Armature inductance ;

$i(t)$ ---- Armature current;

$e_b(t)$ — Generator back-EMF ;

$T(t)$ ---Grenerator torque ;

J ----- **moment of inertia** ;

B ----- Viscous friction coefficient

1) $T(t) = K i(t)$ $T(t)$ ——Torque K ——Torque coefficient

2)
$$e_b(t) = K_b \frac{d\theta(t)}{dt}$$
 $e_b(t)$ ——Back-EMF K_b ——Back-EMF coefficient

3)
$$L \frac{di(t)}{dt} + R i(t) + e_b(t) = e_a(t)$$
 $e_a(t)$ ——Armature Voltage

4)
$$J \frac{d^2\theta(t)}{dt^2} + B \frac{d\theta(t)}{dt} = T(t)$$

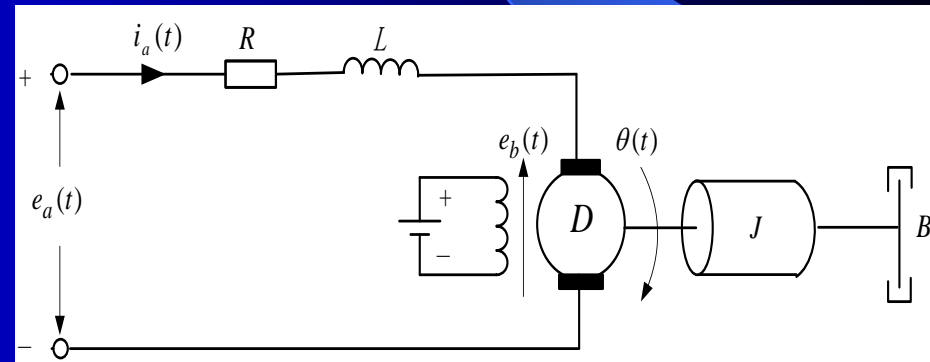
Laplace transform:

1) $T(s) = K I(s)$

2) $E_b(s) = K_b s \theta(s)$

3) $E_a(s) = (L s + R) I(s) + E_b(s)$

4) $T(s) = (J s^2 + B s) \theta(s)$



Elimination Variables $E_b(s)$ 、 $T(s)$ and $I(s)$

$$\frac{\theta(s)}{E_a(s)} = \frac{K}{s[LJs^2 + (LB + RJ)s + (RB + KK_b)]}$$

Input $E_a(s)$, output speed $\Omega(s)$, we can get:

$$\frac{\Omega(s)}{E_a(s)} = \frac{K}{LJs^2 + (LB + RJ)s + (RB + KK_b)}$$

This is a typical **oscillation Element** transfer function
Frequency response:

$$\frac{\Omega(j\omega)}{E_a(j\omega)} = \frac{K}{(RB + KK_b - LJ\omega^2) + j(LB + RJ)\omega}$$

If we ignore the effect of L :

$$\frac{\theta(s)}{E_a(s)} = \frac{K_m}{s(T_m s + 1)}$$

$$\frac{\Omega(s)}{E_a(s)} = \frac{K_m}{T_m s + 1}$$

where: $k_m = K/(RB + KK_b)$ ——— Generator gain constant

$T_m = RJ/(RB + KK_b)$ ——— Generator time constant

When T_m approximation to zero, we can get:

$$\frac{\theta(s)}{E_a(s)} = \frac{1/K_b}{s} = \frac{K'}{s} \quad (K' = \frac{1}{K_b})$$

EX3: Mechanical System

Input-----Force : $f(t)$,

Output----Displacement: $x(t)$

Differential equation:

$$f(t) = M \frac{d^2 x(t)}{dt^2} + B \frac{dx(t)}{dt} + Kx(t)$$

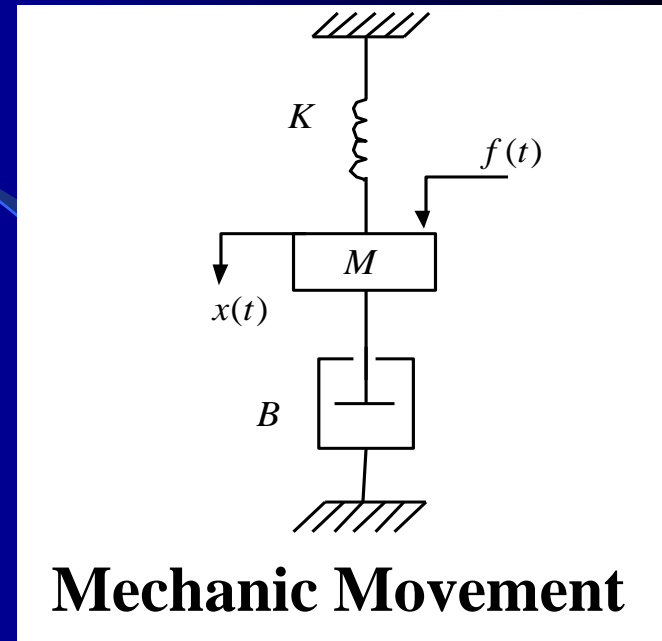
Where: K ——Elastic coefficient

M ——Mass of the object,

B ——Viscous friction coefficient

Transfer function:

$$H(s) = \frac{X(s)}{F(s)} = \frac{\frac{1}{K}}{\frac{M}{K}s^2 + \frac{B}{K}s + 1}$$



6. First-order Differential Element (6. 一阶微分环节)

Features: not only the output related to input itself, but also related to the change rate of input

Dynamic equation:

$$c(t) = T \frac{dr(t)}{dt} + r(t)$$

Transfer function: $H(s) = Ts + 1$

Frequency response: $H(j\omega) = j\omega T + 1$

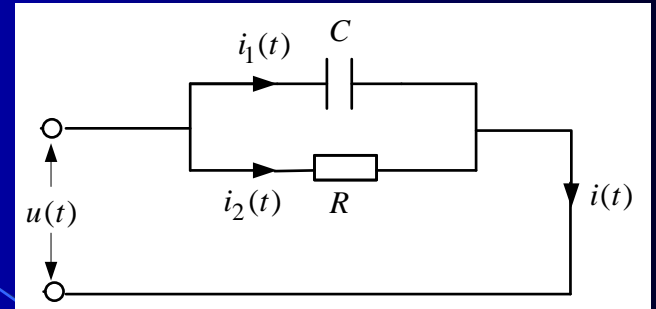
RC Circuit

Input: $u(t)$, **Output:** $i(t)$, hence

$$\begin{aligned} i(t) &= i_1(t) + i_2(t) = C \frac{du(t)}{dt} + \frac{u(t)}{R} \\ &= \frac{1}{R} \left[RC \frac{du(t)}{dt} + u(t) \right] = \tau \frac{du(t)}{dt} + u(t) \end{aligned}$$

Transfer function: $\frac{I(s)}{U(s)} = \tau s + 1$ ($R=1\Omega$ $RC=\tau$)

Frequency response $H(j\omega) = 1 + j\omega \tau$



7. two-order Differential Element

(7.二阶微分环节)

Dynamic equation:

Transfer function:

$$c(t) = T^2 \frac{d^2 r(t)}{dt^2} + 2\zeta T \frac{dr(t)}{dt} + r(t)$$

Frequency function:

$$G(s) = \frac{C(s)}{R(s)} = T^2 s^2 + 2\zeta T s + 1$$

$$\begin{aligned} G(j\omega) &= T^2 (j\omega)^2 + 2\zeta T (j\omega) + 1 \\ &= (1 - T^2 \omega^2) + j2\zeta T \omega \end{aligned}$$

Summary

(1) Different physical system may share the same form of transfer function .

Example: Introduced two oscillation examples, a mechanical system, the other is a electrical system, but they have the same transfer function form.

(2) Transfer function of different form could be get from one system, if we choose different variable as the input and output.

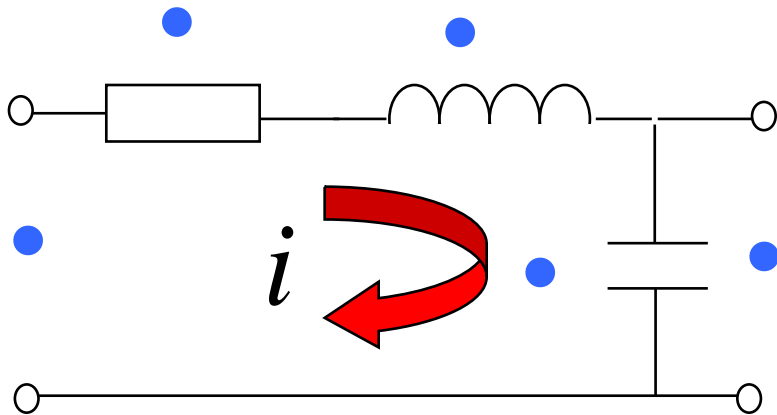
Example:

Capacitance: input-current; output -voltage, it is a **Integral Element**.

vice, input-voltage; output -current, it is a **Differential Element**.

Appendix 1- Other typical System Model

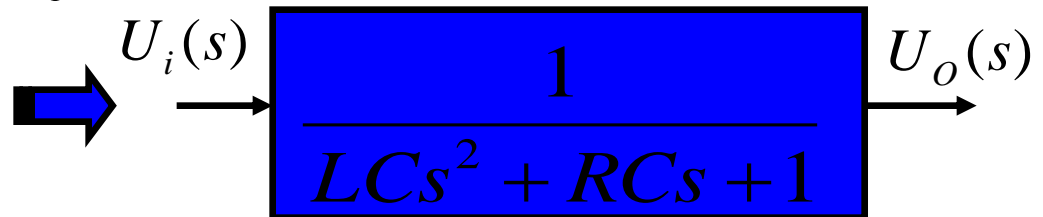
SP1. Electric Circuit System



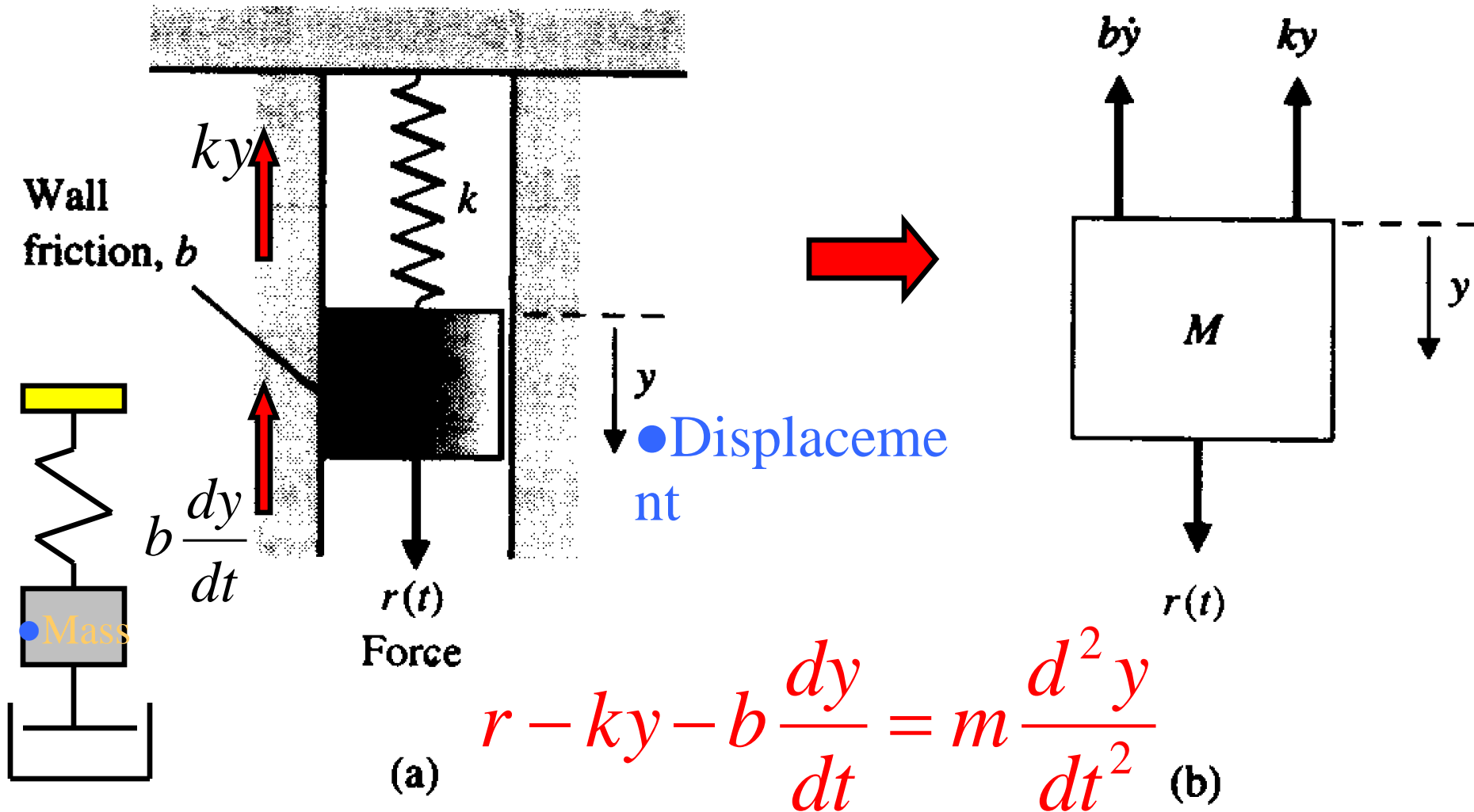
$$u_i = R \cdot i + L \frac{di}{dt} + u_o$$

$$u_o = \frac{1}{C} \int i \, dt$$

$$\Rightarrow LC \frac{d^2 u_o}{dt^2} + RC \frac{du_o}{dt} + u_o = u_i$$



Appendix 2-. Mechanic Movement (Translation) System with Spring – Mass – Damper

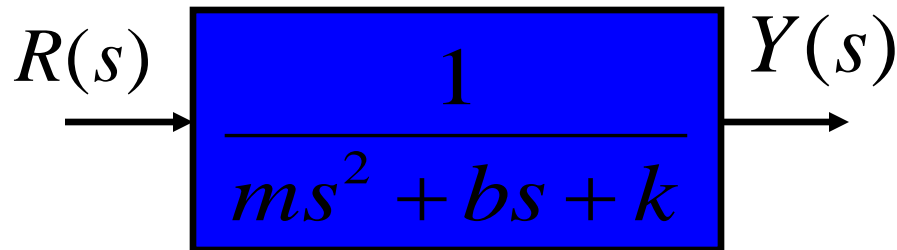


Appendix 2-. Mechanic Movement System with Spring – Mass – Damper

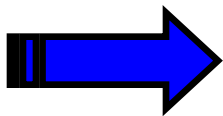
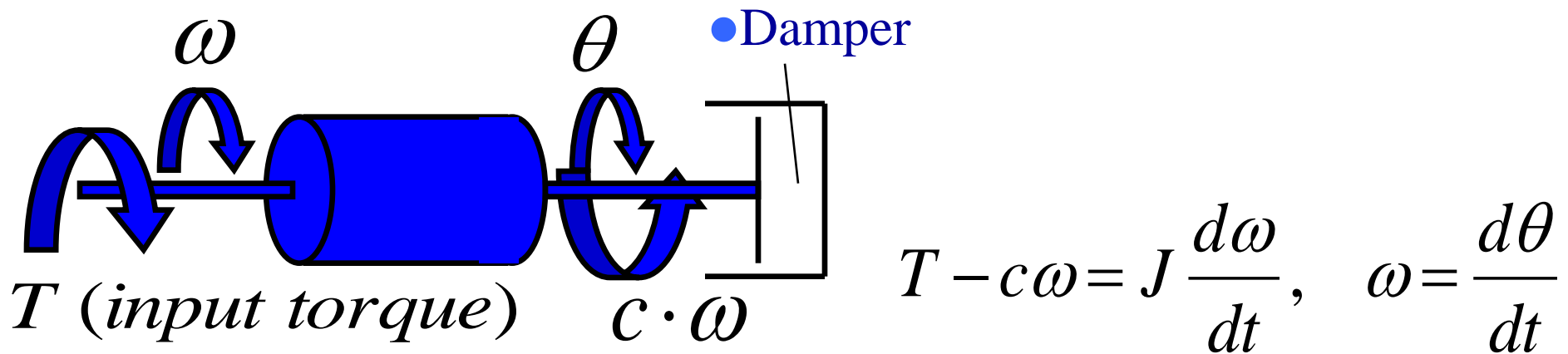
$$r - ky - b \frac{dy}{dt} = m \frac{d^2 y}{dt^2}$$



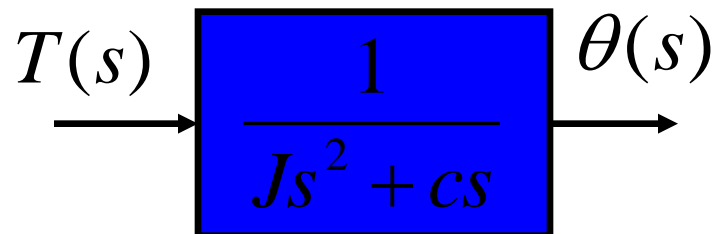
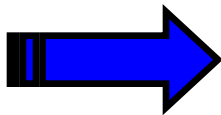
$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = r$$



Appendix -3. Mechanic Rotational System with Damper

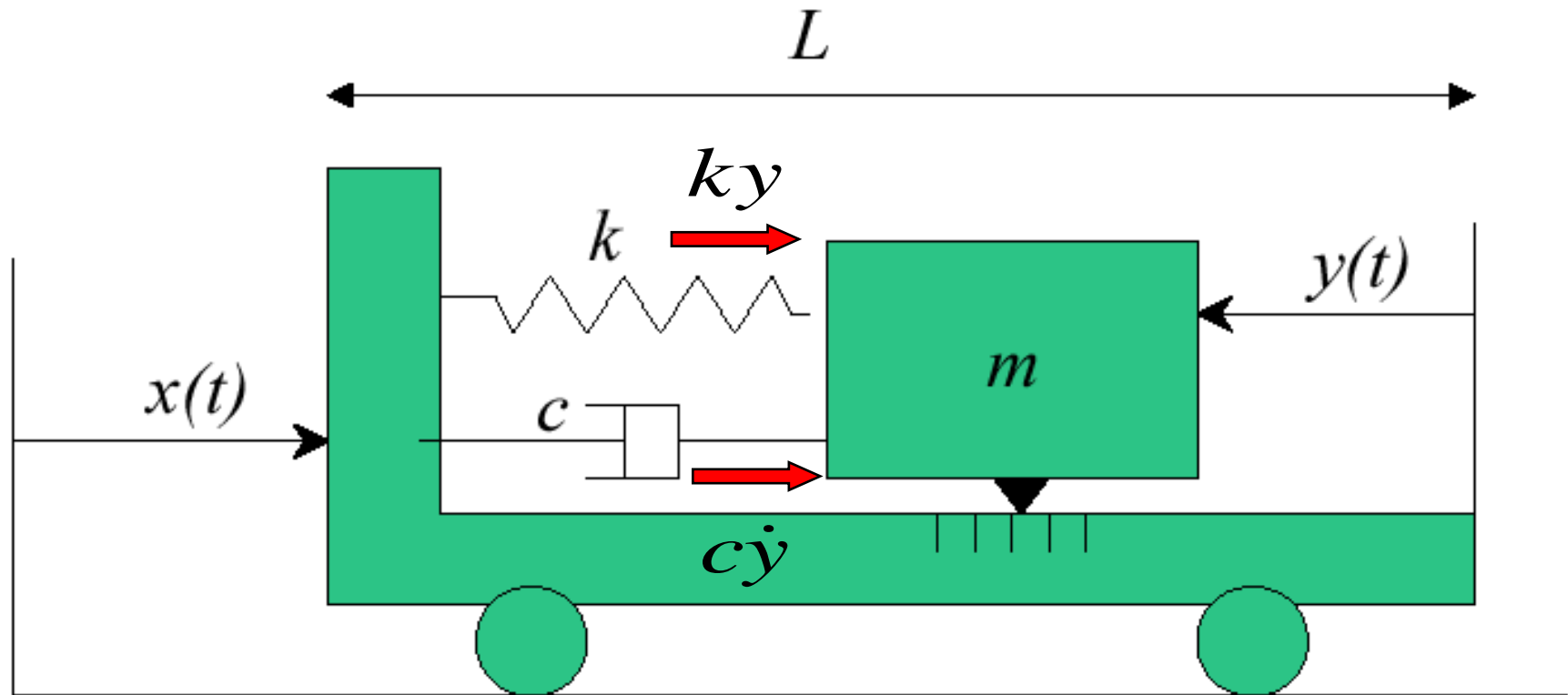


$$J \frac{d^2\theta}{dt^2} + c \frac{d\theta}{dt} = T$$



Appendix 4.

Example (mechanical accelerometer):



Input: \ddot{x}

Output: y

Newton's second law

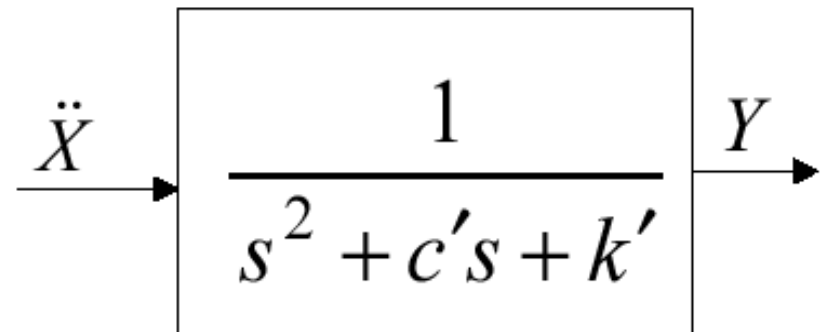
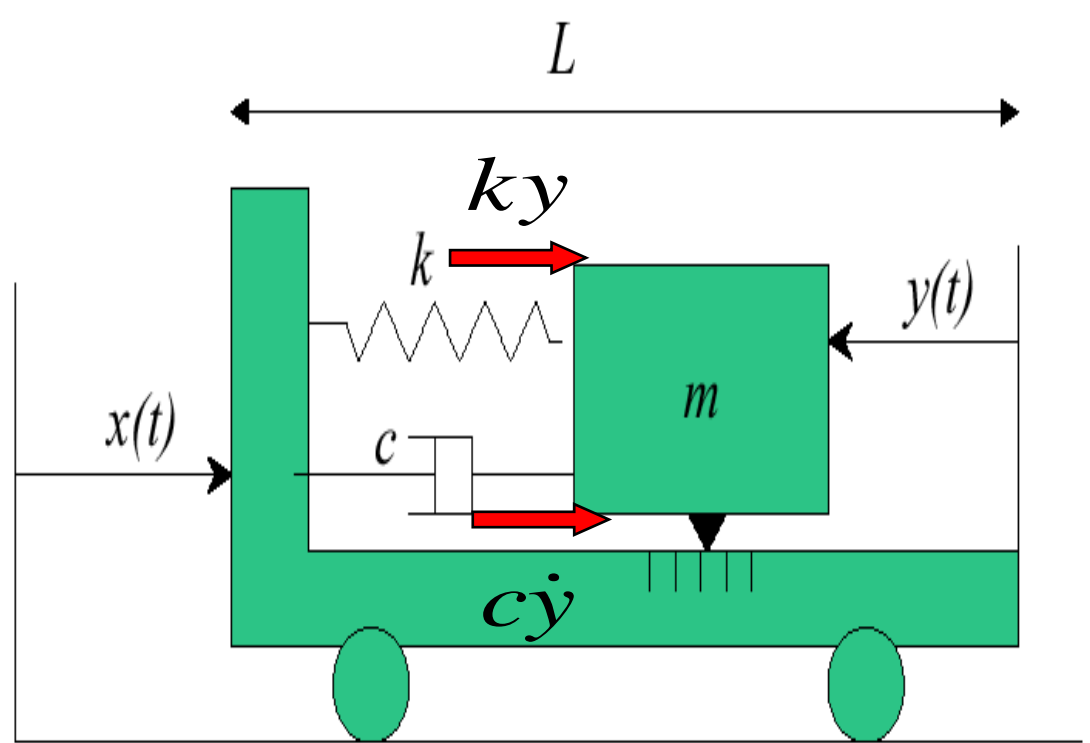
$$m(\ddot{x} - \ddot{y}) = ky + c\dot{y}$$

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = \ddot{x}$$

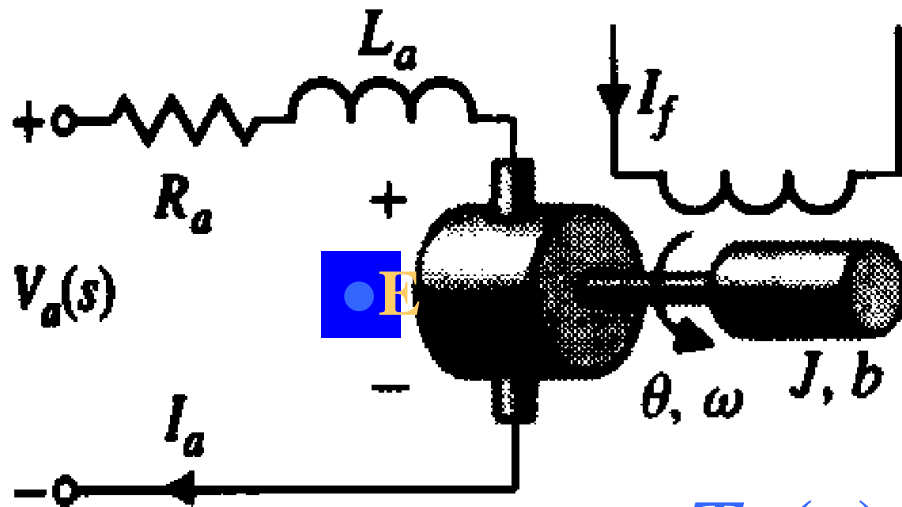
$$\ddot{y} + c'\dot{y} + k'y = \ddot{x}$$

Taking the Laplace transform

$$(s^2 + c's + k')Y = U = \ddot{X}$$



Appendix 5. The armature – controlled DC Motor



$$V_a(s) = E + (R_a + sL_a)I_a(s)$$

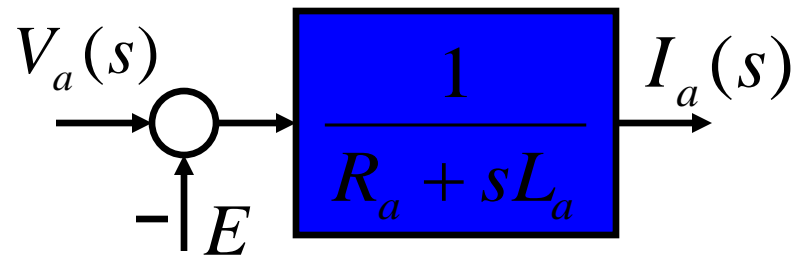
$$T_m(s) = K_m I_a(s)$$

$$E = K_b \omega_s$$

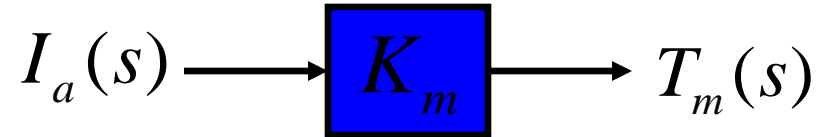
$$T_m(s) - T_d(s) - b\omega(s) = Js\omega(s)$$

$$\omega(s) = s\theta(s)$$

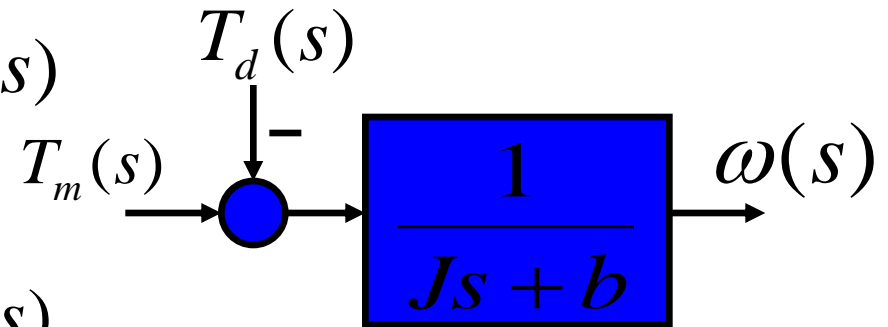
$$V_a(s) = E + (R_a + sL_a)I_a(s)$$



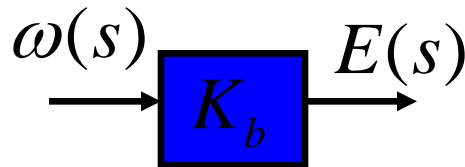
$$T_m(s) = K_m I_a(s)$$



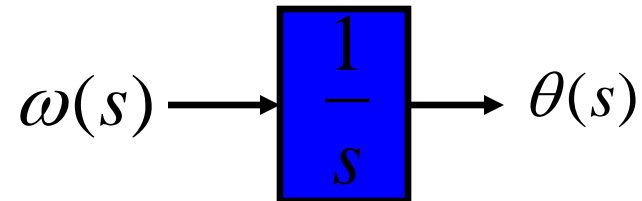
$$T_m(s) - T_d(s) - b\omega(s) = Js\omega(s)$$



$$E = K_b \omega_s$$



$$\omega(s) = s\theta(s)$$



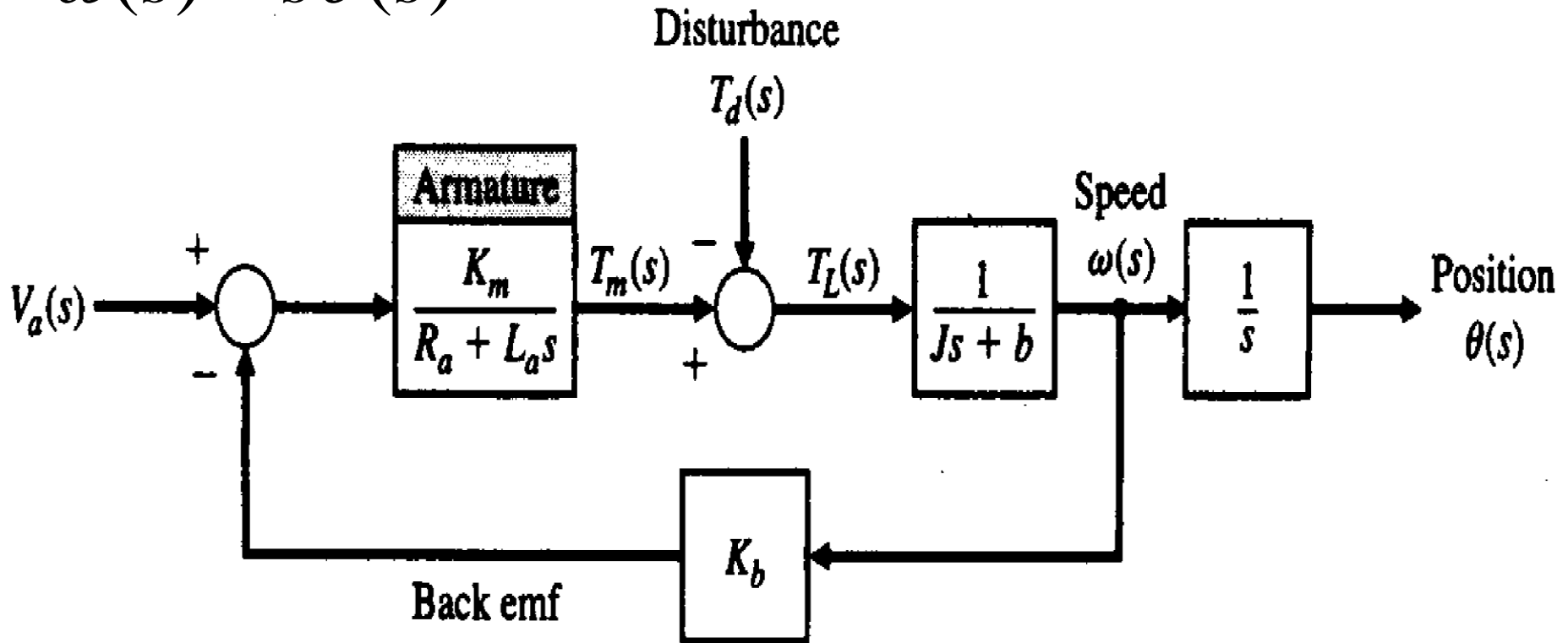
$$V_a(s) = E + (R_a + sL_a)I_a(s)$$

$$T_m(s) = K_m I_a(s)$$

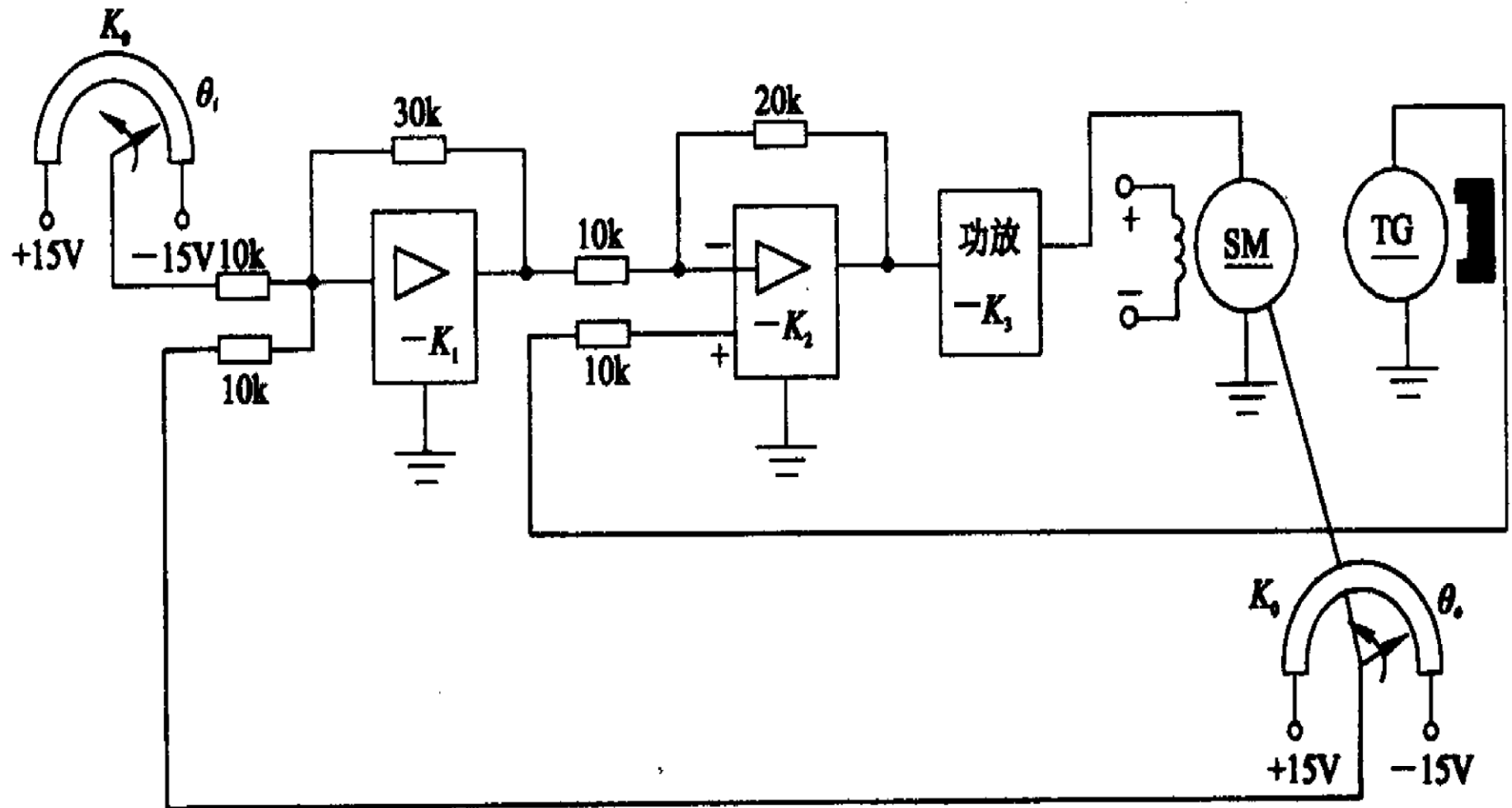
$$E = K_b \omega_s$$

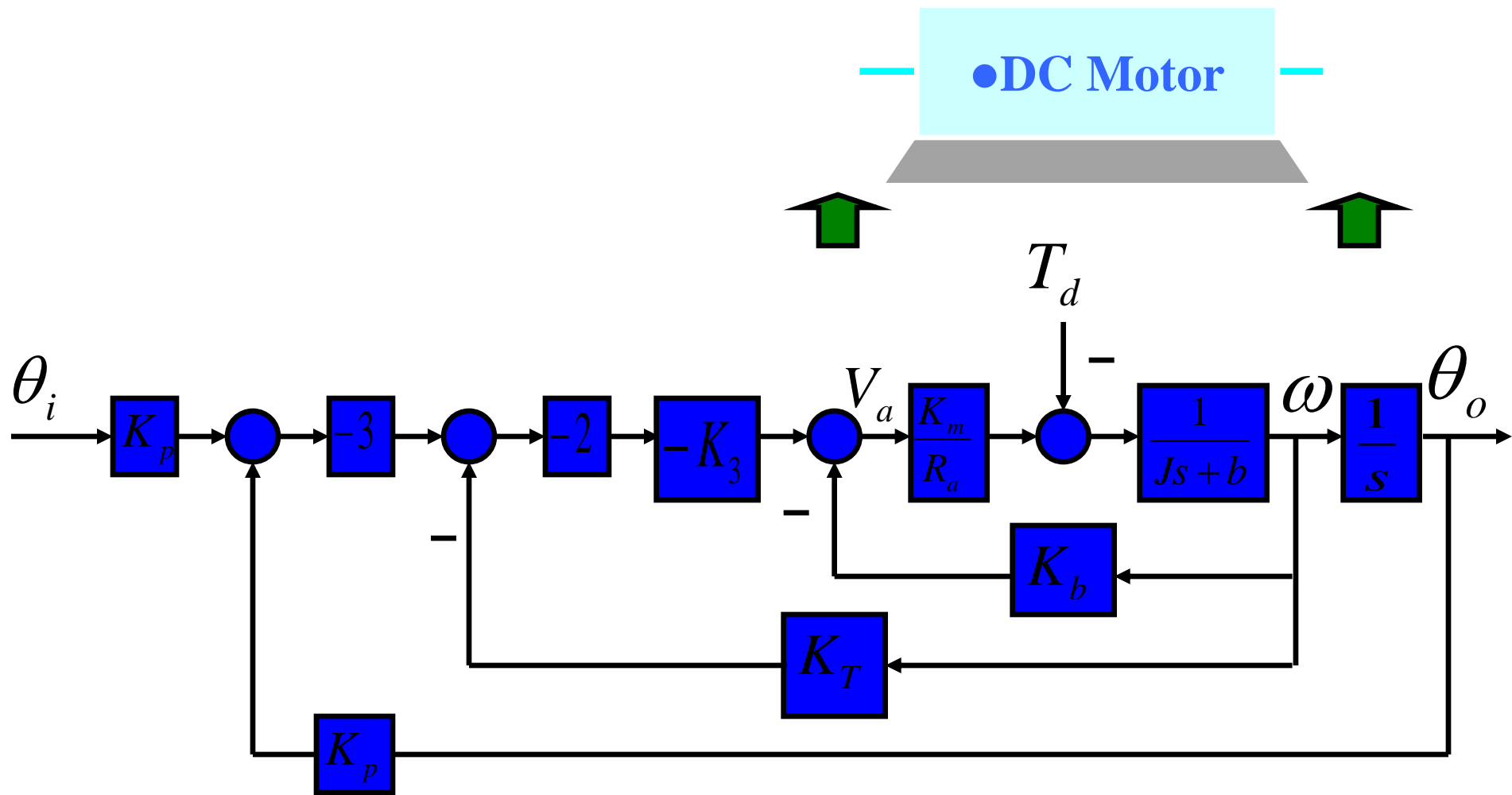
$$T_m(s) - T_d(s) - b\omega(s) = Js\omega(s)$$

$$\omega(s) = s\theta(s)$$



Appendix 6. Complex Control System –Position Tracking System





Homework(1)

1. Page 49 ,End-of-Chapter Questions
 - Q1,Q2,Q3
2. Page 50-51 ,Problems
 - 2.1
 - 2.8
3. Deadline Sep.26.2012

Homework(2)

Reference for Complex function and
Integral transformation

1. 《积分变换》，南京工学院，高等教育出版社；
2. 《工程数学—复变函数》，西安交通大学，人民教育出版社